

Evaluation of the general 3-loop vacuum Feynman integral

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We discuss the systematic evaluation of 3-loop vacuum integrals with arbitrary masses. Using integration by parts, the general integral of this type can be reduced algebraically to a few basis integrals. We define a set of modified finite basis integrals that are particularly convenient for expressing renormalized quantities. The basis integrals can be computed numerically by solving coupled first-order differential equations, using as boundary conditions the analytically known special cases that depend on only one mass scale. We provide the results necessary to carry this out, and introduce an implementation in the form of a public software package called 3VIL (3-loop Vacuum Integral Library), which efficiently computes the numerical values of the basis integrals for any specified masses. 3VIL is written in C, and can be linked from C, C++, or FORTRAN code.

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I. INTRODUCTION

With the discovery of the Higgs boson, the Standard Model has reached a milestone of experimental completion. Because all of the particle masses and couplings are now known directly or indirectly with well-defined experimental precisions, it is worthwhile to extend the calculations of the predictions of the Standard Model, as well as competitor extensions of it, to the kind of accuracy that requires loop integrals to be calculated beyond 2-loop order. In general it is useful to reduce theoretical uncertainties to the level at which they are completely negligible compared to the corresponding experimental and parametric errors. In some cases, the only reliable way to obtain estimates of theoretical error of a given calculation is to compute to an additional order in perturbation theory. In this paper, we address the problem of calculating the general 3-loop vacuum Feynman integral in dimensional regularization, with arbitrary propagator masses.[†]

The computation of 2-loop vacuum integrals with arbitrary masses has been reduced to polylogarithms or equivalent functions, see refs. [2–6]. At 3-loop order, the vacuum integrals with one non-zero mass have been solved [7–13]. In some special cases, 3-loop integrals with two distinct non-zero masses are also known analytically [14–21]. These results are reviewed below in section V, in our notations, and a few new two-scale special cases are added. The program MATAD [22] is available for computations of vacuum diagrams with one non-zero mass scale, and more generally can be used in conjunction with expansions in ratios of squared masses and external momenta (see, for example, [23, 24]).

One obvious application of the results given below is to the 3-loop effective potential of a general theory, with the Standard Model and its supersymmetric extensions as particular cases. In the latter case it is not clear *a priori* what the ordering or hierarchies of the masses will turn out to be. Even in the Standard Model case, it is helpful to be able to perform and present calculations in a way that does not require expansions in particular mass hierarchies. Therefore, in the following we use an approach that does not depend on such expansions. The evaluation of the integrals is performed using the differential equations method [25, 26], [5], [27–37]. Here we use expressions for the derivatives of the basis integrals with respect to their squared mass arguments. The method is implemented in a public open-source computer package called 3VIL (3-loop Vacuum Integral Library), which is structurally similar to, and compatible with, our earlier program TSIL [33] for the calculation of 2-loop self-energy basis integrals.

The rest of this paper is organized as follows. In section II, we establish our notations and conventions, including the basis of 3-loop vacuum integrals to which all others can be reduced using the method of integration by parts [38]. Two related but distinct alternative sets of basis integrals are defined, one incorporating counterterms in such a way that renormalized quantities are efficiently written in terms of them. The relation between the ϵ expansions of the two sets of basis integrals is given in a section III. The derivatives of the basis integrals with respect to the propagator squared masses and the renormalization scale are given in section IV. In section V, we review the known analytical special cases, all of which have only one or two distinct non-zero

[†] A numerical solution of general 3-loop vacuum integrals has also independently been obtained by A. Freitas [1] in a way different from ours, namely in terms of 1-dimensional (or, in the 6-propagator case, 2-dimensional) integral representations, by making use of dispersion relations.

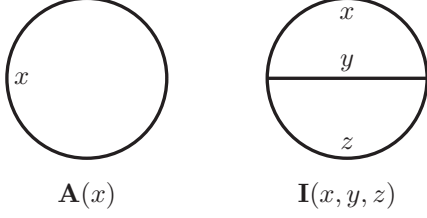


FIG. 2.1: The topologies for the 1-loop and 2-loop basis integrals $\mathbf{A}(x)$ and $\mathbf{I}(x, y, z)$ defined in eqs. (2.3) and (2.4).

masses. Section VI provides the differential equations used to compute the 3-loop vacuum integrals in the more general case of arbitrary masses. Section VII describes the implementation of these results and an introduction to our public and open-source computer program 3VIL. Section VIII contains some concluding remarks.

II. BASIS INTEGRAL DEFINITIONS

In this section, we establish our notational conventions and define the vacuum basis integrals up to 3-loop order. After Wick rotation, loop momentum integrations are carried out in

$$d = 4 - 2\epsilon \quad (2.1)$$

Euclidean dimensions, and denoted by

$$\int_p \equiv \mu^{4-d} \int \frac{d^d p}{(2\pi)^d}, \quad (2.2)$$

where μ is a regularization mass scale. Vacuum Feynman diagrams at 1-loop and 2-loop orders can be written in terms of the basis integrals $\mathbf{A}(x)$ and $\mathbf{I}(x, y, z)$ depicted in Figure 2.1. Here

$$\mathbf{A}(x) = 16\pi^2 \int_p \frac{1}{p^2 + x} = \Gamma(-1 + \epsilon) \left(\frac{4\pi\mu^2}{x} \right)^\epsilon x, \quad (2.3)$$

where x is the propagator squared mass. The two-loop order basis integral is defined by

$$\mathbf{I}(x, y, z) = (16\pi^2)^2 \int_p \int_q \frac{1}{[p^2 + x][q^2 + y][(p - q)^2 + z]}, \quad (2.4)$$

which is symmetric on interchanges of any pair of squared masses x, y, z . Any 2-loop vacuum Feynman diagram can be reduced to sums of \mathbf{I} functions and products of two \mathbf{A} functions, with coefficients that are ratios of polynomials in the squared masses and the spacetime dimension d .

When presenting results for renormalized physical quantities (whether in the $\overline{\text{MS}}$ scheme or any other scheme), it is convenient to eliminate the ϵ -dependent basis functions \mathbf{A} and \mathbf{I} in favor of ϵ -independent functions that include the effects of counterterms. We define

$$\overline{\ln}(x) \equiv \ln(x/Q^2), \quad (2.5)$$

with the $\overline{\text{MS}}$ renormalization scale Q defined by

$$Q^2 = 4\pi e^{-\gamma_E} \mu^2. \quad (2.6)$$

Then we have the expansion:

$$\mathbf{A}(x) = -\frac{x}{\epsilon} + A(x) + \epsilon A_\epsilon(x) + \epsilon^2 A_{\epsilon^2}(x) + \dots, \quad (2.7)$$

where[†]

$$A(x) = x[\overline{\ln}(x) - 1], \quad (2.8)$$

$$A_\epsilon(x) = x \left[-\frac{1}{2} \overline{\ln}^2(x) + \overline{\ln}(x) - 1 - \frac{\pi^2}{12} \right], \quad (2.9)$$

$$A_{\epsilon^2}(x) = x \left[\frac{1}{6} \overline{\ln}^3(x) - \frac{1}{2} \overline{\ln}^2(x) + \left(1 + \frac{\pi^2}{12} \right) \overline{\ln}(x) - 1 - \frac{\pi^2}{12} + \frac{\zeta_3}{3} \right]. \quad (2.10)$$

The ϵ -expansion of the two-loop basis integral can be written as:

$$\mathbf{I}(x, y, z) = \frac{I_2(x, y, z)}{\epsilon^2} + \frac{I_1(x, y, z)}{\epsilon} + I_0(x, y, z) + \epsilon I_\epsilon(x, y, z) + \dots, \quad (2.11)$$

where the pole terms are

$$I_2(x, y, z) = -(x + y + z)/2, \quad (2.12)$$

$$I_1(x, y, z) = A(x) + A(y) + A(z) - (x + y + z)/2. \quad (2.13)$$

However, instead of writing results in terms of I_0 and I_ϵ , it is more convenient to follow[‡] ref. [2] by defining the “renormalized” basis integral:

$$I(x, y, z) = \lim_{\epsilon \rightarrow 0} \left[\mathbf{I}(x, y, z) - I_{\text{div}}^{(1)}(x, y, z) - I_{\text{div}}^{(2)}(x, y, z) \right], \quad (2.14)$$

where the 1-loop and 2-loop ultraviolet (UV) sub-divergences are, respectively,

$$I_{\text{div}}^{(1)}(x, y, z) = \frac{1}{\epsilon} [\mathbf{A}(x) + \mathbf{A}(y) + \mathbf{A}(z)], \quad (2.15)$$

$$I_{\text{div}}^{(2)}(x, y, z) = \frac{1}{2} (x + y + z) \left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \right). \quad (2.16)$$

[†] For brevity, we never include the common scale Q explicitly among the arguments of loop integral functions.

[‡] However, the notation is slightly different; $I(x, y, z)$ in the present paper is equal to $(16\pi^2)^2 \hat{I}(x, y, z)$ in ref. [2].

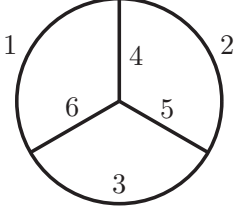


FIG. 2.2: The topology of the scalar Feynman diagram for the general 3-loop vacuum integral $\mathbf{T}^{(n_1, n_2, n_3, n_4, n_5, n_6)}(x_1, x_2, x_3, x_4, x_5, x_6)$ defined in eq. (2.18).

Then one obtains:

$$I(x, y, z) = I_0(x, y, z) - A_\epsilon(x) - A_\epsilon(y) - A_\epsilon(z). \quad (2.17)$$

Now the 2-loop renormalized effective potential can be written efficiently in terms of $I(x, y, z)$ and $A(x)$, as was done in ref. [2] for the Standard Model and ref. [39] for general renormalizable theories, without needing to use the functions I_0 or A_ϵ . For 3-loop renormalized quantities such as the effective potential, it is possible and natural to avoid the use of I_ϵ , and the functions I_0 and A_ϵ only appear in the combination I . The integrals $I(x, y, z)$ and $I_0(x, y, z)$ and $I_\epsilon(x, y, z)$ can be evaluated in terms of polylogarithms, using the methods of ref. [2]. For completeness, these results are listed in section V below.

A general 3-loop order vacuum Feynman diagram will involve scalar integrals of the form shown in Figure 2.2:

$$\mathbf{T}^{(n_1, n_2, n_3, n_4, n_5, n_6)}(x_1, x_2, x_3, x_4, x_5, x_6) = (16\pi^2)^3 \int_{p_1} \int_{p_2} \int_{p_3} \frac{1}{[p_1^2 + x_1]^{n_1} [p_2^2 + x_2]^{n_2} [p_3^2 + x_3]^{n_3} [(p_1 - p_2)^2 + x_4]^{n_4} [(p_2 - p_3)^2 + x_5]^{n_5} [(p_3 - p_1)^2 + x_6]^{n_6}}, \quad (2.18)$$

where the propagator powers n_i can be positive, negative, or 0. These integrals satisfy identities involving interchanges of the pairs (n_i, x_i) , as implied by the tetrahedral symmetry of the graphical representation shown in Figure 2.2. They also satisfy 9 identities implied by integration by parts [10, 22, 38]:

$$0 = \int_{p_1} \int_{p_2} \int_{p_3} \frac{\partial}{\partial p_i^\mu} [p_j^\mu X] \quad (2.19)$$

for $i, j = 1, 2, 3$, where X is any product of propagators as in eq. (2.18). The identities for $(i, j) = (1, 1)$ and $(1, 2)$ can be written as, acting on eq. (2.18),

$$0 = d - 2n_1 - n_4 - n_6 + 2x_1 \mathbf{1}^+ n_1 + (x_1 - x_2 + x_4 - \mathbf{1}^- + \mathbf{2}^-) \mathbf{4}^+ n_4 + (x_1 - x_3 + x_6 - \mathbf{1}^- + \mathbf{3}^-) \mathbf{6}^+ n_6, \quad (2.20)$$

$$0 = n_4 - n_1 + (x_1 + x_2 - x_4 - \mathbf{2}^- + \mathbf{4}^-) \mathbf{1}^+ n_1 + (x_1 - x_2 - x_4 - \mathbf{1}^- + \mathbf{2}^-) \mathbf{4}^+ n_4 + (x_1 - x_3 - x_4 + x_5 - \mathbf{1}^- + \mathbf{3}^- + \mathbf{4}^- - \mathbf{5}^-) \mathbf{6}^+ n_6, \quad (2.21)$$

and there are $2+5=7$ other independent ones that can be obtained from the above two as permutations implied by the tetrahedral symmetry. Here, the bold-faced raising and lowering operators

are defined to increase or decrease the power of the corresponding propagator:

$$\mathbf{j}^\pm \mathbf{T}^{(\dots, n_j, \dots)} = \mathbf{T}^{(\dots, n_j \pm 1, \dots)}. \quad (2.22)$$

The dimensional analysis identity

$$0 = 3d/2 + \sum_{j=1}^6 (x_j \mathbf{j}^+ - 1) n_j. \quad (2.23)$$

can also be obtained by combining the three integration by parts identities that involve d . Equation (2.20), and each of 11 identities (not all independent) obtained by permutations of it using the symmetries of the tetrahedron, is an example of what is sometimes known as the triangle rule.

By repeated application of these integration by parts identities, any 3-loop vacuum integral \mathbf{T} can eventually be reduced to a linear combination of integrals from a basis set, with coefficients that are ratios of polynomials in d and the squared masses. The integrals in the basis are of five types, and can be defined as:

$$\mathbf{H}(u, v, w, x, y, z) = \mathbf{T}^{(1,1,1,1,1,1)}(u, v, w, x, y, z), \quad (2.24)$$

$$\mathbf{G}(w, u, z, v, y) = \mathbf{T}^{(1,1,1,0,1,1)}(u, v, w, x, y, z), \quad (2.25)$$

$$\mathbf{F}(u, v, y, z) = \mathbf{T}^{(2,1,0,0,1,1)}(u, v, w, x, y, z), \quad (2.26)$$

$$\mathbf{A}(u)\mathbf{I}(v, w, y) = \mathbf{T}^{(1,1,1,0,1,0)}(u, v, w, x, y, z), \quad (2.27)$$

$$\mathbf{A}(u)\mathbf{A}(v)\mathbf{A}(w) = \mathbf{T}^{(1,1,1,0,0,0)}(u, v, w, x, y, z), \quad (2.28)$$

and the integrals obtained by permutations of the arguments of these according to the symmetries of the tetrahedron. The last two are products of lower loop integrals, and therefore present no problems. The integral defined by

$$\mathbf{E}(u, v, y, z) = \mathbf{T}^{(1,1,0,0,1,1)}(u, v, w, x, y, z) \quad (2.29)$$

is useful, as we will see below, but it is not part of this canonical basis, because it can be reduced to the \mathbf{F} integrals using the linear algebraic identity

$$\mathbf{E}(u, v, y, z) = [u\mathbf{F}(u, v, y, z) + v\mathbf{F}(v, u, y, z) + y\mathbf{F}(y, u, v, z) + z\mathbf{F}(z, u, v, y)] / (-2 + 3\epsilon), \quad (2.30)$$

which follows from dimensional analysis. Note that

$$\mathbf{F}(u, v, y, z) = -\frac{\partial}{\partial u} \mathbf{E}(u, v, y, z), \quad (2.31)$$

which allows some identities satisfied by the \mathbf{F} integrals to be more easily derived or succinctly written in terms of the \mathbf{E} integrals. The graph topologies associated with the functions \mathbf{H} , \mathbf{G} , \mathbf{F} , and \mathbf{E} are shown in Figure 2.3. We note that the relation of our notations and conventions to those

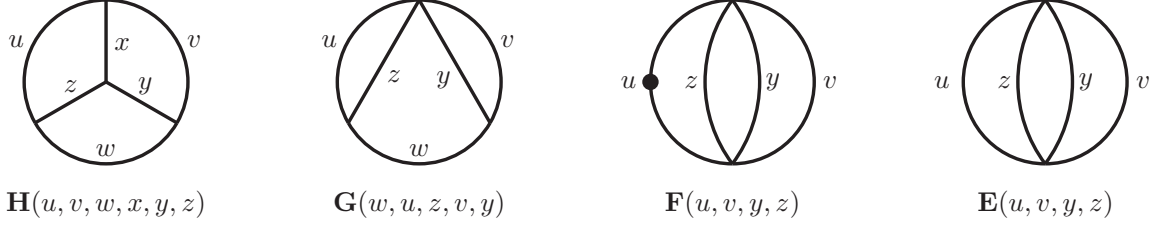


FIG. 2.3: The topologies for the integrals $\mathbf{H}(u, v, w, x, y, z)$ and $\mathbf{G}(w, u, z, v, y)$ and $\mathbf{F}(u, v, y, z)$ and $\mathbf{E}(u, v, y, z)$ defined in eqs. (2.24), (2.25), (2.26), and (2.29), respectively. The dot on the diagram for $\mathbf{F}(u, v, y, z)$ is used to indicate a squared propagator. A complete basis for the 3-loop vacuum integrals consists of integrals of the types \mathbf{H} , \mathbf{G} , \mathbf{F} , and the products of 1-loop and 2-loop integrals \mathbf{A} and \mathbf{I} . The integral \mathbf{E} is a useful adjunct, but is not included in the basis due to its redundancy, because of eq. (2.30).

of the functions U_n defined in [1] is given by:

$$(Q^2)^{-3\epsilon} \mathbf{F}(u, v, y, z) = -U_4(u, v, y, z), \quad (2.32)$$

$$(Q^2)^{-3\epsilon} \mathbf{G}(w, u, z, v, y) = -U_5(u, z, v, y, w), \quad (2.33)$$

$$(Q^2)^{-3\epsilon} \mathbf{H}(u, v, w, x, y, z) = U_6(u, v, y, z, x, w), \quad (2.34)$$

where the $\overline{\text{MS}}$ renormalization scale Q is defined by eq. (2.6).

It is again useful to define ϵ -independent modified basis integrals that will appear in renormalized quantities written in their most succinct forms. This is done by subtracting UV sub-divergences and then taking the 4-dimensional limit. For the \mathbf{E} and \mathbf{F} integrals, we define:

$$E(u, v, y, z) = \lim_{\epsilon \rightarrow 0} \left[\mathbf{E}(u, v, y, z) - E_{\text{div}}^{(1)}(u, v, y, z) - E_{\text{div}}^{(2)}(u, v, y, z) - E_{\text{div}}^{(3)}(u, v, y, z) \right], \quad (2.35)$$

where the 1-loop, 2-loop, and 3-loop UV sub-divergences are, respectively,

$$E_{\text{div}}^{(1)}(u, v, y, z) = \frac{1}{\epsilon} \mathbf{A}(u) \mathbf{A}(v) + (5 \text{ permutations}), \quad (2.36)$$

$$E_{\text{div}}^{(2)}(u, v, y, z) = \left[\frac{1}{2\epsilon^2} (v + y + z) + \frac{1}{2\epsilon} \left(\frac{u}{2} - v - y - z \right) \right] \mathbf{A}(u) + (3 \text{ permutations}), \quad (2.37)$$

$$E_{\text{div}}^{(3)}(u, v, y, z) = \left[\frac{1}{3\epsilon^3} - \frac{2}{3\epsilon^2} + \frac{1}{3\epsilon} \right] (uv + uy + uz + vy + vz + yz) \\ + \left[\frac{1}{6\epsilon^2} - \frac{3}{8\epsilon} \right] (u^2 + v^2 + y^2 + z^2). \quad (2.38)$$

Then, renormalized quantities can be written in terms of the function

$$F(u, v, y, z) = -\frac{\partial}{\partial u} E(u, v, y, z). \quad (2.39)$$

From eq. (2.30) and the other definitions above, one finds the linear algebraic expression of the

redundancy of E :

$$\begin{aligned}
E(u, v, y, z) = & \frac{1}{2} \left[-uF(u, v, y, z) - vF(v, u, y, z) - yF(y, u, v, z) - zF(z, u, v, y) \right. \\
& + A(u)A(v) + A(u)A(y) + A(u)A(z) + A(v)A(y) + A(v)A(z) + A(y)A(z) \\
& + (u/2 - v - y - z)A(u) + (v/2 - u - y - z)A(v) \\
& + (y/2 - u - v - z)A(y) + (z/2 - u - v - y)A(z) \\
& \left. + uv + uy + uz + vy + vz + yz - 9(u^2 + v^2 + y^2 + z^2)/8 \right]. \tag{2.40}
\end{aligned}$$

However, the function $F(u, v, y, z)$ has a logarithmic infrared divergence in the limit $u \rightarrow 0$. Therefore, we further define:

$$\overline{F}(u, v, y, z) \equiv F(u, v, y, z) + \overline{\ln}(u)I(v, y, z), \tag{2.41}$$

which is well-defined for all finite values of its squared mass arguments. Some of the results described below are given in terms of the modified basis function \overline{F} , and the program library 3VIL uses \overline{F} rather than F internally, but both functions are available as outputs, and eq. (2.41) can of course be used to translate between the F and \overline{F} functions whenever necessary. In expressions below, we will use whichever of F or \overline{F} is more convenient.

Similarly, we define the modified basis function:

$$\begin{aligned}
G(w, u, z, v, y) = & \lim_{\epsilon \rightarrow 0} \left[\mathbf{G}(w, u, z, v, y) - G_{\text{div}}^{(1)}(w, u, z, v, y) - G_{\text{div}}^{(2)}(w, u, z, v, y) \right. \\
& \left. - G_{\text{div}}^{(3)}(w, u, z, v, y) \right], \tag{2.42}
\end{aligned}$$

where the 1-loop, 2-loop, and 3-loop UV sub-divergences are

$$G_{\text{div}}^{(1)}(w, u, z, v, y) = \frac{1}{\epsilon} [\mathbf{I}(w, u, z) + \mathbf{I}(w, v, y)], \tag{2.43}$$

$$G_{\text{div}}^{(2)}(w, u, z, v, y) = \left(-\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} \right) [\mathbf{A}(u) + \mathbf{A}(v) + \mathbf{A}(y) + \mathbf{A}(z)] - \frac{1}{\epsilon^2} \mathbf{A}(w), \tag{2.44}$$

$$G_{\text{div}}^{(3)}(w, u, z, v, y) = \left(-\frac{1}{6\epsilon^3} + \frac{1}{2\epsilon^2} - \frac{2}{3\epsilon} \right) (u + v + y + z) + \left(-\frac{1}{3\epsilon^3} + \frac{1}{3\epsilon^2} + \frac{1}{3\epsilon} \right) w. \tag{2.45}$$

A useful aspect of the definition eq. (2.42) is that when renormalized expressions are written in terms of G rather than \mathbf{G} , then one does not need to use the ϵ^1 parts of the expansions of \mathbf{I} functions; only I functions are necessary.

Finally, the \mathbf{H} function is free of 1-loop and 2-loop UV sub-divergences, but does have a 3-loop UV sub-divergence. Therefore we define:

$$H(u, v, w, x, y, z) = \lim_{\epsilon \rightarrow 0} \left[\mathbf{H}(u, v, w, x, y, z) - H_{\text{div}}^{(3)}(u, v, w, x, y, z) \right] \tag{2.46}$$

where

$$H_{\text{div}}^{(3)}(u, v, w, x, y, z) = 2\zeta_3/\epsilon. \quad (2.47)$$

The function $H(u, v, w, x, y, z)$ is finite [except in the case $u = v = w = x = y = z = 0$ where it has an infrared logarithmic divergence; see any one of eqs. (5.52)-(5.61) below].

By use of the integration by parts identities, the evaluation of a general 3-loop Feynman vacuum diagram is thus reduced to the problem of computing $I(x, y, z)$, $\overline{F}(u, v, y, z)$, $G(w, u, z, v, y)$, and $H(u, v, w, x, y, z)$. Although renormalized quantities are most efficiently written in terms of these quantities rather than their bold-faced counterparts $\mathbf{I}(x, y, z)$, $\mathbf{F}(u, v, y, z)$, $\mathbf{G}(w, u, z, v, y)$, and $\mathbf{H}(u, v, w, x, y, z)$, the formulas for the ϵ -expansions of the latter are provided in the next section.

The 2-loop integral $I(x, y, z)$ is known in terms of dilogarithms, but in general $\overline{F}(u, v, y, z)$, $G(w, u, z, v, y)$, and $H(u, v, w, x, y, z)$ cannot be done analytically in terms of polylogarithms or other simple functions. Therefore, numerical methods are necessary.

III. EXPANSIONS IN ϵ FOR THE INTEGRALS \mathbf{E} , \mathbf{F} , \mathbf{G} , AND \mathbf{H}

The ϵ expansions of the \mathbf{E} , \mathbf{F} , \mathbf{G} , and \mathbf{H} integrals can be written in the forms:

$$\begin{aligned} \mathbf{E}(u, v, y, z) = & \frac{1}{\epsilon^3} E_3(u, v, y, z) + \frac{1}{\epsilon^2} E_2(u, v, y, z) + \frac{1}{\epsilon} E_1(u, v, y, z) \\ & + E_0(u, v, y, z) + \dots \end{aligned} \quad (3.1)$$

$$\begin{aligned} \mathbf{F}(u, v, y, z) = & \frac{1}{\epsilon^3} F_3(u, v, y, z) + \frac{1}{\epsilon^2} F_2(u, v, y, z) + \frac{1}{\epsilon} F_1(u, v, y, z) \\ & + F_0(u, v, y, z) + \dots, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \mathbf{G}(w, u, z, v, y) = & \frac{1}{\epsilon^3} G_3(w, u, z, v, y) + \frac{1}{\epsilon^2} G_2(w, u, z, v, y) + \frac{1}{\epsilon} G_1(w, u, z, v, y) \\ & + G_0(w, u, z, v, y) + \dots \end{aligned} \quad (3.3)$$

$$\mathbf{H}(u, v, w, x, y, z) = \frac{1}{\epsilon} H_1(u, v, w, x, y, z) + H_0(u, v, w, x, y, z) + \dots \quad (3.4)$$

Using the formulas in section II, one obtains:

$$E_3(u, v, y, z) = (uv + uy + uz + vy + vz + yz)/3, \quad (3.5)$$

$$\begin{aligned} E_2(u, v, y, z) = & -[(v + y + z)A(u) + (u + y + z)A(v) + (u + v + z)A(y) + (u + v + y)A(z)]/2 \\ & + (uv + uy + uz + vy + vz + yz)/3 - (u^2 + v^2 + y^2 + z^2)/12, \end{aligned} \quad (3.6)$$

$$\begin{aligned} E_1(u, v, y, z) = & A(u)A(v) + A(u)A(y) + A(u)A(z) + A(v)A(y) + A(v)A(z) + A(y)A(z) \\ & - (v + y + z)[A_\epsilon(u) + A(u)]/2 - (u + y + z)[A_\epsilon(v) + A(v)]/2 \\ & - (u + v + z)[A_\epsilon(y) + A(y)]/2 - (u + v + y)[A_\epsilon(z) + A(z)]/2 \\ & + [uA(u) + vA(v) + yA(y) + zA(z)]/4 \\ & + (uv + uy + uz + vy + vz + yz)/3 - 3(u^2 + v^2 + y^2 + z^2)/8, \end{aligned} \quad (3.7)$$

$$E_0(u, v, y, z) = E(u, v, y, z) + A(u)[A_\epsilon(v) + A_\epsilon(y) + A_\epsilon(z)] + A(v)[A_\epsilon(u) + A_\epsilon(y) + A_\epsilon(z)]$$

$$\begin{aligned}
& +A(y)[A_\epsilon(u) + A_\epsilon(v) + A_\epsilon(z)] + A(z)[A_\epsilon(u) + A_\epsilon(v) + A_\epsilon(y)] \\
& - (v + y + z)[A_\epsilon(u) + A_{\epsilon^2}(u)]/2 - (u + y + z)[A_\epsilon(v) + A_{\epsilon^2}(v)]/2 \\
& - (u + v + z)[A_\epsilon(y) + A_{\epsilon^2}(y)]/2 - (u + v + y)[A_\epsilon(z) + A_{\epsilon^2}(z)]/2 \\
& + [uA_\epsilon(u) + vA_\epsilon(v) + yA_\epsilon(y) + zA_\epsilon(z)]/4.
\end{aligned} \tag{3.8}$$

Then one can use

$$F_n(u, v, y, z) = -\frac{\partial}{\partial u} E_n(u, v, y, z) \tag{3.9}$$

for $n = 0, 1, 2, 3$, which can be evaluated using

$$\frac{\partial}{\partial u} A(u) = A(u)/u + 1, \tag{3.10}$$

$$\frac{\partial}{\partial u} A_\epsilon(u) = [A_\epsilon(u) - A(u)]/u, \tag{3.11}$$

$$\frac{\partial}{\partial u} A_{\epsilon^2}(u) = [A_{\epsilon^2}(u) - A_\epsilon(u)]/u, \tag{3.12}$$

with the results:

$$F_3(u, v, y, z) = -(v + y + z)/3, \tag{3.13}$$

$$F_2(u, v, y, z) = (v + y + z)A(u)/2u + [A(v) + A(y) + A(z)]/2 + (u + v + y + z)/6, \tag{3.14}$$

$$\begin{aligned}
F_1(u, v, y, z) = & -[A(v) + A(y) + A(z)]A(u)/u + (v + y + z)A_\epsilon(u)/2u \\
& + [A_\epsilon(v) + A_\epsilon(y) + A_\epsilon(z) - A(u) - A(v) - A(y) - A(z)]/2 + u/2 + (v + y + z)/6,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
F_0(u, v, y, z) = & F(u, v, y, z) + (v + y + z)A_{\epsilon^2}(u)/2u - [A(v) + A(y) + A(z)]A_\epsilon(u)/u \\
& + [A(v) + A(y) + A(z) - A_\epsilon(v) - A_\epsilon(y) - A_\epsilon(z) + u/4 - v/2 - y/2 - z/2]A(u)/u \\
& + [A_{\epsilon^2}(v) + A_{\epsilon^2}(y) + A_{\epsilon^2}(z) - A_\epsilon(u) - A_\epsilon(v) - A_\epsilon(y) - A_\epsilon(z)]/2.
\end{aligned} \tag{3.16}$$

Similarly, we obtain:

$$G_3(w, u, z, v, y) = -(2w + u + v + y + z)/6, \tag{3.17}$$

$$G_2(w, u, z, v, y) = [A(u) + A(v) + A(y) + A(z) - u - v - y - z]/2 + A(w) - 2w/3, \tag{3.18}$$

$$\begin{aligned}
G_1(w, u, z, v, y) = & I(u, w, z) + I(v, w, y) + A_\epsilon(w) + [A_\epsilon(u) + A_\epsilon(v) + A_\epsilon(y) + A_\epsilon(z) \\
& + A(u) + A(v) + A(y) + A(z)]/2 + (w - 2u - 2v - 2y - 2z)/3,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
G_0(w, u, z, v, y) = & G(w, u, z, v, y) + I_\epsilon(u, w, z) + I_\epsilon(v, w, y) - A_{\epsilon^2}(w) + [A_\epsilon(u) + A_\epsilon(v) \\
& + A_\epsilon(y) + A_\epsilon(z) - A_{\epsilon^2}(u) - A_{\epsilon^2}(v) - A_{\epsilon^2}(y) - A_{\epsilon^2}(z)]/2.
\end{aligned} \tag{3.20}$$

Finally,

$$H_1(u, v, w, x, y, z) = 2\zeta(3), \tag{3.21}$$

$$H_0(u, v, w, x, y, z) = H(u, v, w, x, y, z). \tag{3.22}$$

Note that the ϵ -independent terms in the expansions, $E_0(u, v, y, z)$ and $F_0(u, v, y, z)$ and $G_0(w, u, z, v, y)$, are not the same things as the more useful functions $E(u, v, y, z)$ and $F(u, v, y, z)$ and $G(w, u, z, v, y)$. The latter appear in renormalized quantities when put into the simplest forms.

IV. DERIVATIVES OF THE BASIS FUNCTIONS

In this section, we give the derivatives of the basis functions defined in the section II with respect to the squared mass arguments and the renormalization scale Q . These can be obtained using the integration by parts identities, and are special cases of the general fact that any vacuum integral can be reduced to the basis. Note that the derivatives of \mathbf{E} and E functions are trivial, in the sense that they are just given by \mathbf{F} and F functions, respectively.

We start with the results in terms of the bold-faced integrals \mathbf{A} , \mathbf{I} , \mathbf{F} , \mathbf{G} , and \mathbf{H} . For the 1-loop and 2-loop order basis integrals,

$$\frac{\partial}{\partial x} \mathbf{A}(x) = (d/2 - 1) \mathbf{A}(x)/x, \quad (4.1)$$

$$\begin{aligned} \frac{\partial}{\partial x} \mathbf{I}(x, y, z) = & \left\{ (d-3)(x-y-z) \mathbf{I}(x, y, z) + (d-2) \left[(x-y+z) \mathbf{A}(x) \mathbf{A}(y)/2x \right. \right. \\ & \left. \left. + (x+y-z) \mathbf{A}(x) \mathbf{A}(z)/2x - \mathbf{A}(y) \mathbf{A}(z) \right] \right\} / \lambda(x, y, z), \end{aligned} \quad (4.2)$$

where

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \quad (4.3)$$

The derivatives of $\mathbf{I}(x, y, z)$ with respect to y and z follow from symmetry.

For the 3-loop basis integrals, the results are more complicated, so that only the structural forms will be shown in print here, with the complete explicit expressions relegated to an ancillary electronic file called `derivatives.txt`, which is included with the arXiv source of this paper. In all cases, the derivatives can be written as:

$$\sum_i k_i X_i \quad (4.4)$$

where X_i are basis integrals, and k_i are rational functions of the squared masses and the spacetime dimension d . In the cases of

$$\frac{\partial}{\partial u} \mathbf{F}(u, v, y, z) \quad (4.5)$$

and

$$\frac{\partial}{\partial v} \mathbf{F}(u, v, y, z), \quad (4.6)$$

the basis integrals appearing in the sum are:

$$X_i = \{ \mathbf{F}(u, v, y, z), \mathbf{F}(v, u, y, z), \mathbf{F}(y, u, v, z), \mathbf{F}(z, u, v, y), \mathbf{A}(u)\mathbf{A}(v)\mathbf{A}(y), \\ \mathbf{A}(u)\mathbf{A}(v)\mathbf{A}(z), \mathbf{A}(u)\mathbf{A}(y)\mathbf{A}(z), \mathbf{A}(v)\mathbf{A}(y)\mathbf{A}(z) \}. \quad (4.7)$$

The derivatives $\frac{\partial}{\partial y}\mathbf{F}(u, v, y, z)$ and $\frac{\partial}{\partial z}\mathbf{F}(u, v, y, z)$ follow from $\frac{\partial}{\partial v}\mathbf{F}(u, v, y, z)$ by symmetry. The denominators of the coefficients k_i in these derivatives contain factors of

$$\begin{aligned} \psi(u, v, y, z) \equiv & u^4 + v^4 + y^4 + z^4 - 4u^3(v + y + z) - 4v^3(u + y + z) - 4y^3(u + v + z) \\ & - 4z^3(u + v + y) + 4u^2(vy + vz + yz) + 4v^2(uy + uz + yz) \\ & + 4y^2(uv + uz + vz) + 4z^2(uy + uv + vy) + 6u^2v^2 + 6u^2y^2 + 6u^2z^2 \\ & + 6v^2y^2 + 6v^2z^2 + 6y^2z^2 - 40uvyz. \end{aligned} \quad (4.8)$$

In the cases of

$$\frac{\partial}{\partial w}\mathbf{G}(w, u, z, v, y) \quad (4.9)$$

and

$$\frac{\partial}{\partial u}\mathbf{G}(w, u, z, v, y), \quad (4.10)$$

the basis integrals in the sum are:

$$X_i = \{ \mathbf{G}(w, u, z, v, y), \mathbf{F}(u, v, y, z), \mathbf{F}(v, u, y, z), \mathbf{F}(y, u, v, z), \mathbf{F}(z, u, v, y), \\ \mathbf{A}(v)\mathbf{I}(w, u, z), \mathbf{A}(y)\mathbf{I}(w, u, z), \mathbf{A}(u)\mathbf{I}(w, v, y), \mathbf{A}(z)\mathbf{I}(w, v, y) \}. \quad (4.11)$$

The denominators of the coefficients for $\frac{\partial}{\partial w}\mathbf{G}(w, u, z, v, y)$ contain factors of $\lambda(u, w, z)$ and $\lambda(v, w, y)$, while the denominators in $\frac{\partial}{\partial u}\mathbf{G}(w, u, z, v, y)$ contain only the factor $\lambda(u, w, z)$. The derivatives $\frac{\partial}{\partial z}\mathbf{G}(w, u, z, v, y)$, $\frac{\partial}{\partial v}\mathbf{G}(w, u, z, v, y)$, and $\frac{\partial}{\partial y}\mathbf{G}(w, u, z, v, y)$ follow from $\frac{\partial}{\partial u}\mathbf{G}(w, u, z, v, y)$ using symmetry.

Finally, in the case of

$$\frac{\partial}{\partial u}\mathbf{H}(u, v, w, x, y, z), \quad (4.12)$$

the necessary basis integrals are:

$$\begin{aligned} X_i = & \{ \mathbf{H}(u, v, w, x, y, z), \mathbf{G}(u, v, x, w, z), \mathbf{G}(v, u, x, w, y), \mathbf{G}(w, u, z, v, y), \\ & \mathbf{G}(x, u, v, y, z), \mathbf{G}(y, v, w, x, z), \mathbf{G}(z, u, w, x, y), \mathbf{F}(u, v, y, z), \mathbf{F}(u, w, x, y), \\ & \mathbf{F}(v, u, y, z), \mathbf{F}(v, w, x, z), \mathbf{F}(w, u, x, y), \mathbf{F}(w, v, x, z), \mathbf{F}(x, u, w, y), \\ & \mathbf{F}(x, v, w, z), \mathbf{F}(y, u, v, z), \mathbf{F}(y, u, w, x), \mathbf{F}(z, u, v, y), \mathbf{F}(z, v, w, x), \end{aligned}$$

$$\begin{aligned}
& \mathbf{A}(w)\mathbf{I}(u, v, x), \mathbf{A}(y)\mathbf{I}(u, v, x), \mathbf{A}(z)\mathbf{I}(u, v, x), \mathbf{A}(v)\mathbf{I}(u, w, z), \\
& \mathbf{A}(x)\mathbf{I}(u, w, z), \mathbf{A}(y)\mathbf{I}(u, w, z), \mathbf{A}(u)\mathbf{I}(v, w, y), \mathbf{A}(x)\mathbf{I}(v, w, y), \\
& \mathbf{A}(z)\mathbf{I}(v, w, y), \mathbf{A}(u)\mathbf{I}(x, y, z), \mathbf{A}(v)\mathbf{I}(x, y, z), \mathbf{A}(w)\mathbf{I}(x, y, z)\}.
\end{aligned} \tag{4.13}$$

The denominators of the coefficients for $\frac{\partial}{\partial u}\mathbf{H}(u, v, w, x, y, z)$ contain factors of $\lambda(u, v, x)$ and $\lambda(u, w, z)$ and

$$\begin{aligned}
\chi(u, v, w, x, y, z) = & u^2y + v^2z + w^2x + x^2w + y^2u + z^2v + uvx - uwx - vwx \\
& - uvy - uwy + vwy - uxy - wxy - uvz + uwz - vwz \\
& - vxz - wxz - uyz - vyz + xyz.
\end{aligned} \tag{4.14}$$

The derivatives of $\mathbf{H}(u, v, w, x, y, z)$ with respect to the other arguments follow from the tetrahedral symmetry.

The corresponding derivatives of the functions A, I, F, \overline{F}, G , and H can be obtained straightforwardly from the results above and the formulas in the previous sections, by expanding in ϵ . The results are quite complicated, so again they are not presented in print here, but are given explicitly in the ancillary file `derivatives.txt`. Where the denominator factors mentioned above vanish, the differential equations governing the basis functions have pseudo-thresholds, but the basis functions themselves are well-defined and smooth for all non-negative u, v, w, x, y, z .

It is also useful to have derivatives with respect to the renormalization scale, for example to check the renormalization group invariance of a calculation of the 3-loop effective potential. Here we present results in terms of the renormalized integrals $A, I, E, F, \overline{F}, G$, and H . For the 1-loop and 2-loop integrals, one finds:

$$Q^2 \frac{\partial}{\partial Q^2} A(x) = -x, \tag{4.15}$$

$$Q^2 \frac{\partial}{\partial Q^2} I(x, y, z) = A(x) + A(y) + A(z) - x - y - z. \tag{4.16}$$

For the 4-propagator 3-loop integrals, we find:

$$\begin{aligned}
Q^2 \frac{\partial}{\partial Q^2} E(u, v, y, z) = & A(u)A(v) + A(u)A(y) + A(u)A(z) + A(v)A(y) + A(v)A(z) + A(y)A(z) \\
& + (u/2 - v - y - z)A(u) + (v/2 - u - y - z)A(v) \\
& + (y/2 - u - v - z)A(y) + (z/2 - u - v - y)A(z) \\
& + uv + uy + uz + vy + vz + yz - 9(u^2 + v^2 + y^2 + z^2)/8,
\end{aligned} \tag{4.17}$$

$$Q^2 \frac{\partial}{\partial Q^2} F(u, v, y, z) = [v + y + z - u - A(v) - A(y) - A(z)]A(u)/u + 7u/4, \tag{4.18}$$

$$Q^2 \frac{\partial}{\partial Q^2} \overline{F}(u, v, y, z) = A(v) + A(y) + A(z) - A(u) - I(v, y, z) - v - y - z + 7u/4. \tag{4.19}$$

For the 5- and 6-propagator 3-loop integrals, we obtain:

$$Q^2 \frac{\partial}{\partial Q^2} G(w, u, z, v, y) = I(w, u, z) + I(w, v, y) + A(u) + A(v) + A(y) + A(z) - 2u - 2v - 2y - 2z + w, \quad (4.20)$$

$$Q^2 \frac{\partial}{\partial Q^2} H(u, v, w, x, y, z) = 6\zeta_3. \quad (4.21)$$

V. KNOWN ANALYTICAL CASES

For some special cases, it is possible to give analytical expressions in closed form for the basis integrals, in terms of the polylogarithm functions $\text{Li}_n(z)$ of complex argument [40]. Although individual terms in expressions below are sometimes complex numbers, the basis vacuum integrals are always real when the squared masses are non-negative. Besides the usual transcendental numbers such as $\ln(2)$, π , ζ_3 and $\text{Li}_4(1/2)$, some expressions below involve the log-sine definite integrals:

$$\text{Ls}_2 \equiv \text{Ls}_2(2\pi/3) = - \int_0^{2\pi/3} dx \ln[2 \sin(x/2)] \approx 0.6766277376064358, \quad (5.1)$$

$$\text{Ls}_3 \equiv \text{Ls}_3(2\pi/3) = - \int_0^{2\pi/3} dx \ln^2[2 \sin(x/2)] \approx -2.1447672125694944, \quad (5.2)$$

$$\text{Ls}'_4 \equiv \text{Ls}_4^{(1)}(2\pi/3) = - \int_0^{2\pi/3} dx x \ln^2[2 \sin(x/2)] \approx -0.4976755516066472. \quad (5.3)$$

The 2-loop vacuum integral basis function $I(x, y, z)$ is well-known, in various cosmetically different but equivalent forms [2–6]. For $z \geq x, y$:

$$\begin{aligned} I(x, y, z) = & s \left[\text{Li}_2(k_1) + \text{Li}_2(k_2) - \ln(k_1) \ln(k_2) + \frac{1}{2} \ln(x/z) \ln(y/z) - \pi^2/6 \right] \\ & + \frac{1}{2}(z - x - y) \overline{\ln}(x) \overline{\ln}(y) + \frac{1}{2}(y - x - z) \overline{\ln}(x) \overline{\ln}(z) + \frac{1}{2}(x - y - z) \overline{\ln}(y) \overline{\ln}(z) \\ & + 2x \overline{\ln}(x) + 2y \overline{\ln}(y) + 2z \overline{\ln}(z) - \frac{5}{2}(x + y + z) \end{aligned} \quad (5.4)$$

where

$$s = \sqrt{\lambda(x, y, z)}, \quad (5.5)$$

$$k_1 = (x + z - y - s)/2z, \quad (5.6)$$

$$k_2 = (y + z - x - s)/2z. \quad (5.7)$$

The cases with $y \geq x, z$ or $x \geq y, z$ are obtained by permuting the arguments of eq. (5.4). Some useful special cases are:

$$I(0, y, z) = (y - z) \left[\text{Li}_2(1 - y/z) + \frac{1}{2} \overline{\ln}^2(z) \right] - y \overline{\ln}(y) \overline{\ln}(z)$$

$$+2y\overline{\ln}(y) + 2z\overline{\ln}(z) - \frac{5}{2}(y+z), \quad (5.8)$$

$$I(x, x, x) = x \left[-\frac{15}{2} + 3\sqrt{3}\text{Li}_2 + 6\overline{\ln}(x) - \frac{3}{2}\overline{\ln}^2(x) \right], \quad (5.9)$$

$$I(0, x, x) = x \left[-5 + 4\overline{\ln}(x) - \overline{\ln}^2(x) \right], \quad (5.10)$$

$$I(0, 0, x) = x \left[-\frac{5}{2} - \frac{\pi^2}{6} + 2\overline{\ln}(x) - \frac{1}{2}\overline{\ln}^2(x) \right], \quad (5.11)$$

$$I(0, 0, 0) = 0. \quad (5.12)$$

Now the results for $I_0(x, y, z)$ can be obtained easily from eqs. (2.9) and (2.17).

The result for $I_\epsilon(x, y, z)$ can be obtained as a straightforward application of the method in ref. [2], and has been given in a more compact form in eqs. (15)-(21) and (41) of ref. [4], based on functions defined in eqs. (11), (12), and (29) of ref. [41]. (See also ref. [42] for the expansion of $\mathbf{I}(x, y, z)$ to all orders in ϵ .) These results for $I_\epsilon(x, y, z)$ take different forms depending on whether $x+y$ is greater or less than z . However, the results can be rewritten in a unified way for all $z \geq x, y$ with $s \neq 0$ and $z \neq x+y$, as:

$$\begin{aligned} I_\epsilon(x, y, z) = & [3 - \overline{\ln}(x) - \overline{\ln}(y)]I(x, y, z) + [x\overline{\ln}^3(x) + y\overline{\ln}^3(y) + z\overline{\ln}^3(z)]/6 \\ & + [x\overline{\ln}^2(x) + y\overline{\ln}^2(y) - 3z\overline{\ln}^2(z)]/2 + [(y-x-z)/4]\overline{\ln}(x)\overline{\ln}(z)\ln(x/z) \\ & + [(x-y-z)/4]\overline{\ln}(y)\overline{\ln}(z)\ln(y/z) + [(z-x-y)/4]\overline{\ln}(x)\overline{\ln}(y)[\overline{\ln}(x) + \overline{\ln}(y)] \\ & + [2x + 2y - z\overline{\ln}(z)]\overline{\ln}(x)\overline{\ln}(y) + 2z\overline{\ln}(z)[\overline{\ln}(x) + \overline{\ln}(y)] + (\pi^2/6 + 1)z\overline{\ln}(z) \\ & + (\pi^2/6 - 3/2)[x\overline{\ln}(x) + y\overline{\ln}(y)] - 5[(y+z)\overline{\ln}(x) + (x+z)\overline{\ln}(y)]/2 \\ & + (\zeta_3/3 - \pi^2/4)(x+y+z) + s \left\{ \text{Li}_3(1-r_x) + \text{Li}_3(1-r_y) + \text{Li}_3(1-r_z) \right. \\ & - \text{Li}_3(1-1/r_x) - \text{Li}_3(1-1/r_y) - \text{Li}_3(1-1/r_z) + \ln(z/x)\text{Li}_2(1-r_x) \\ & + \ln(z/y)\text{Li}_2(1-r_y) + \frac{1}{4}\ln(r_x)\ln(r_y)\ln(r_z) + \frac{1}{4}\ln(z/x)\ln(r_x)\ln(xr_x/z) \\ & + \frac{1}{4}\ln(z/y)\ln(r_y)\ln(yr_y/z) + \frac{\eta}{4}[\ln^2(-s^2/xy) - \ln(r_x)\ln(r_y) \\ & \left. + [\ln(r_x) + \ln(r_y) - \ln(r_z)]^2/4] \right\}, \end{aligned} \quad (5.13)$$

where s was defined above in eq. (5.5), and

$$r_x = (s+x-y-z)^2/4yz, \quad (5.14)$$

$$r_y = (s+y-x-z)^2/4xz, \quad (5.15)$$

$$r_z = (s+z-x-y)^2/4xy, \quad (5.16)$$

which implies that $r_x r_y r_z = 1$, and

$$\eta \equiv \ln(r_x) + \ln(r_y) + \ln(r_z) = \begin{cases} -2\pi i & (\text{for } x+y < z), \\ 0 & (\text{for } x+y > z). \end{cases} \quad (5.17)$$

The special case with $s = 0$ is obtained by simply removing all of the terms multiplied by s (i.e., the ones enclosed in curly brackets) in eq. (5.13). The special case $z = x + y$ can be computed by taking the limit $z \rightarrow x + y$ of eq. (5.13), either from above or from below; these limits coincide, despite the branch cut discontinuity in eq. (5.17). Other mass orderings $x \geq y, z$ or $y \geq x, z$ are obtained by permuting the arguments of eq. (5.13). Some useful special cases are:

$$\begin{aligned} I_\epsilon(0, x, y) = & (y - x) \left\{ \text{Li}_3(1 - x/y) - \text{Li}_3(1 - y/x) + [\overline{\ln}(x) + \overline{\ln}(y) - 3] \text{Li}_2(1 - x/y) \right\} \\ & + (y/6) \overline{\ln}^3(x) + (x - y/2) \overline{\ln}^2(x) \overline{\ln}(y) + (y/2) \overline{\ln}(x) \overline{\ln}^2(y) \\ & + (y/2 - x/3) \overline{\ln}^3(y) - (3x/2) \overline{\ln}^2(x) - 3x \overline{\ln}(x) \overline{\ln}(y) + (3x/2 - 3y) \overline{\ln}^2(y) \\ & + (7 + \pi^2/6) [x \overline{\ln}(x) + y \overline{\ln}(y)] + (\zeta_3/3 - 15/2 - \pi^2/4) (x + y), \end{aligned} \quad (5.18)$$

$$\begin{aligned} I_\epsilon(x, x, x) = & x \left\{ 2 \overline{\ln}^3(x) - 9 \overline{\ln}^2(x) + [21 + \pi^2/2 - 6\sqrt{3}\text{Ls}_2] \overline{\ln}(x) - 45/2 - 3\pi^2/4 \right. \\ & \left. + [3 - \ln(3)] 3\sqrt{3}\text{Ls}_2 + 3\sqrt{3}\text{Ls}_3 + \zeta_3 + \pi^3/2\sqrt{3} \right\}, \end{aligned} \quad (5.19)$$

$$I_\epsilon(0, x, x) = x \left[\frac{4}{3} \overline{\ln}^3(x) - 6 \overline{\ln}^2(x) + \left(14 + \frac{\pi^2}{3} \right) \overline{\ln}(x) - 15 - \frac{\pi^2}{2} + \frac{2\zeta_3}{3} \right], \quad (5.20)$$

$$I_\epsilon(0, 0, x) = x \left[\frac{2}{3} \overline{\ln}^3(x) - 3 \overline{\ln}^2(x) + \left(7 + \frac{\pi^2}{2} \right) \overline{\ln}(x) - \frac{15}{2} - \frac{3\pi^2}{4} + \frac{4\zeta_3}{3} \right], \quad (5.21)$$

$$I_\epsilon(0, 0, 0) = 0. \quad (5.22)$$

The results for the 3-loop integrals **E**, **F**, **G**, **H** involving propagators that are either massless or contain a single non-zero mass scale were obtained in [7–13]. A particularly useful and systematic source for them is found in [13]. For convenience, we provide below these results in terms of our modified functions E, F, \overline{F}, G, H . The expansions of **E**, **F**, **G**, **H** up through order ϵ^0 can be reconstructed from these results, using the results of section III of the present paper.

The special cases involving four propagators with all propagator squared masses equal to either 0 or x include:

$$E(0, 0, 0, 0) = 0, \quad (5.23)$$

$$E(0, 0, 0, x) = x^2 \left[-\frac{133}{48} - \frac{\pi^2}{12} + \frac{13}{8} \overline{\ln}(x) - \frac{1}{4} \overline{\ln}^2(x) \right], \quad (5.24)$$

$$E(0, 0, x, x) = x^2 \left[\frac{8\zeta_3}{3} - \frac{89}{24} - \frac{3}{4} \overline{\ln}(x) + \frac{3}{2} \overline{\ln}^2(x) - \frac{1}{3} \overline{\ln}^3(x) \right], \quad (5.25)$$

$$E(0, x, x, x) = x^2 \left[\frac{9\sqrt{3}}{2} \text{Ls}_2 - \frac{45}{16} - \frac{57}{8} \overline{\ln}(x) + \frac{21}{4} \overline{\ln}^2(x) - \overline{\ln}^3(x) \right], \quad (5.26)$$

$$E(x, x, x, x) = x^2 \left[-\frac{1}{12} - \frac{35}{2} \overline{\ln}(x) + 11 \overline{\ln}^2(x) - 2 \overline{\ln}^3(x) \right], \quad (5.27)$$

and

$$F(x, 0, 0, 0) = x \left[\frac{47}{12} + \frac{\pi^2}{6} - \frac{11}{4} \overline{\ln}(x) + \frac{1}{2} \overline{\ln}^2(x) \right], \quad (5.28)$$

$$F(x, 0, 0, x) = x \left[\frac{49}{12} - \frac{8}{3}\zeta_3 - \frac{3}{4}\overline{\ln}(x) - \overline{\ln}^2(x) + \frac{1}{3}\overline{\ln}^3(x) \right], \quad (5.29)$$

$$F(x, 0, x, x) = x \left[\frac{17}{4} - 3\sqrt{3}\text{Ls}_2 + \frac{5}{4}\overline{\ln}(x) - \frac{5}{2}\overline{\ln}^2(x) + \frac{2}{3}\overline{\ln}^3(x) \right], \quad (5.30)$$

$$F(x, x, x, x) = x \left[\frac{53}{12} + \frac{13}{4}\overline{\ln}(x) - 4\overline{\ln}^2(x) + \overline{\ln}^3(x) \right], \quad (5.31)$$

and

$$\overline{F}(0, 0, 0, 0) = 0, \quad (5.32)$$

$$\overline{F}(0, 0, 0, x) = x \left[\frac{1}{6} + \frac{\pi^2}{6} - \frac{2}{3}\zeta_3 - \left(\frac{1}{2} + \frac{\pi^2}{6} \right) \overline{\ln}(x) + \frac{1}{2}\overline{\ln}^2(x) - \frac{1}{6}\overline{\ln}^3(x) \right], \quad (5.33)$$

$$\overline{F}(0, 0, x, x) = x \left[\frac{1}{3} + \frac{8}{3}\zeta_3 - \overline{\ln}(x) + \overline{\ln}^2(x) - \frac{1}{3}\overline{\ln}^3(x) \right], \quad (5.34)$$

$$\overline{F}(x, 0, 0, 0) = x \left[\frac{47}{12} + \frac{\pi^2}{6} - \frac{11}{4}\overline{\ln}(x) + \frac{1}{2}\overline{\ln}^2(x) \right], \quad (5.35)$$

$$\overline{F}(x, 0, 0, x) = x \left[\frac{49}{12} - \frac{8}{3}\zeta_3 - \left(\frac{13}{4} + \frac{\pi^2}{6} \right) \overline{\ln}(x) + \overline{\ln}^2(x) - \frac{1}{6}\overline{\ln}^3(x) \right], \quad (5.36)$$

$$\overline{F}(x, 0, x, x) = x \left[\frac{17}{4} - 3\sqrt{3}\text{Ls}_2 - \frac{15}{4}\overline{\ln}(x) + \frac{3}{2}\overline{\ln}^2(x) - \frac{1}{3}\overline{\ln}^3(x) \right], \quad (5.37)$$

$$\overline{F}(x, x, x, x) = x \left[\frac{53}{12} + \left(3\sqrt{3}\text{Ls}_2 - \frac{17}{4} \right) \overline{\ln}(x) + 2\overline{\ln}^2(x) - \frac{1}{2}\overline{\ln}^3(x) \right], \quad (5.38)$$

and others obtained by permutations implied by the symmetries of the graphs. There is only one such case for which we do not know an exact analytic expression:

$$\overline{F}(0, x, x, x) \approx x \left[9.09686753726327768 \dots + (3\sqrt{3}\text{Ls}_2 - 3/2)\overline{\ln}(x) + \frac{3}{2}\overline{\ln}^2(x) - \frac{1}{2}\overline{\ln}^3(x) \right]. \quad (5.39)$$

Here, the numerical part was found using high-order series solutions of the differential equation. The cases with five or six propagators that are all the same or 0 are:

$$G(0, 0, 0, 0, 0) = 0, \quad (5.40)$$

$$G(0, 0, 0, 0, x) = x \left[-\frac{15}{2} - \frac{\pi^2}{2} + \frac{2}{3}\zeta_3 + \left(\frac{11}{2} + \frac{\pi^2}{6} \right) \overline{\ln}(x) - \frac{3}{2}\overline{\ln}^2(x) + \frac{1}{6}\overline{\ln}^3(x) \right], \quad (5.41)$$

$$G(x, 0, 0, 0, 0) = x \left[-\frac{7}{3} - \frac{2\pi^2}{3} - \frac{2}{3}\zeta_3 + \left(4 + \frac{\pi^2}{3} \right) \overline{\ln}(x) - 2\overline{\ln}^2(x) + \frac{1}{3}\overline{\ln}^3(x) \right], \quad (5.42)$$

$$G(0, 0, 0, x, x) = x \left[-15 - \frac{8}{3}\zeta_3 + 11\overline{\ln}(x) - 3\overline{\ln}^2(x) + \frac{1}{3}\overline{\ln}^3(x) \right], \quad (5.43)$$

$$G(0, 0, x, 0, x) = x \left[-15 - \frac{\pi^2}{3} + \frac{16}{3}\zeta_3 + \left(11 + \frac{\pi^2}{3} \right) \overline{\ln}(x) - 3\overline{\ln}^2(x) + \frac{1}{3}\overline{\ln}^3(x) \right], \quad (5.44)$$

$$G(x, 0, 0, 0, x) = x \left[-\frac{59}{6} - \frac{\pi^2}{2} + \left(\frac{19}{2} + \frac{\pi^2}{6} \right) \overline{\ln}(x) - \frac{7}{2}\overline{\ln}^2(x) + \frac{1}{2}\overline{\ln}^3(x) \right], \quad (5.45)$$

$$G(0, 0, x, x, x) = x \left[-\frac{45}{2} + 9\sqrt{3}\text{Ls}_2 + \left(\frac{33}{2} + \frac{\pi^2}{6} \right) \overline{\ln}(x) - \frac{9}{2}\overline{\ln}^2(x) + \frac{1}{2}\overline{\ln}^3(x) \right], \quad (5.46)$$

$$G(x, 0, 0, x, x) = x \left[-\frac{52}{3} + 6\sqrt{3}\text{Ls}_2 - \frac{\pi^2}{3} - \frac{2\pi^3}{9\sqrt{3}} - \frac{4}{3}\zeta_3 + \left(15 + \frac{\pi^2}{6} - 3\sqrt{3}\text{Ls}_2 \right) \overline{\ln}(x) - 5\overline{\ln}^2(x) + \frac{2}{3}\overline{\ln}^3(x) \right], \quad (5.47)$$

$$G(x, 0, x, 0, x) = x \left[-\frac{52}{3} + \frac{8}{3}\zeta_3 + 15\overline{\ln}(x) - 5\overline{\ln}^2(x) + \frac{2}{3}\overline{\ln}^3(x) \right], \quad (5.48)$$

$$G(0, x, x, x, x) = x \left[-30 + \frac{56}{3}\zeta_3 + 22\overline{\ln}(x) - 6\overline{\ln}^2(x) + \frac{2}{3}\overline{\ln}^3(x) \right], \quad (5.49)$$

$$G(x, 0, x, x, x) = x \left[-\frac{149}{6} + 9\sqrt{3}\text{Ls}_2 + \left(\frac{41}{2} - 3\sqrt{3}\text{Ls}_2 \right) \overline{\ln}(x) - \frac{13}{2}\overline{\ln}^2(x) + \frac{5}{6}\overline{\ln}^3(x) \right], \quad (5.50)$$

$$G(x, x, x, x, x) = x \left[-\frac{97}{3} + 12\sqrt{3}\text{Ls}_2 + 6\zeta_3 + \left(26 - 6\sqrt{3}\text{Ls}_2 \right) \overline{\ln}(x) - 8\overline{\ln}^2(x) + \overline{\ln}^3(x) \right], \quad (5.51)$$

and

$$H(0, 0, 0, 0, 0, x) = \frac{\pi^4}{30} + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.52)$$

$$H(0, 0, 0, 0, x, x) = -\frac{\pi^4}{18} + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.53)$$

$$H(0, x, 0, 0, 0, x) = 16\text{Li}_4(1/2) - \frac{7\pi^4}{60} + \frac{2}{3}\ln^2(2)[\ln^2(2) - \pi^2] + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.54)$$

$$H(0, 0, 0, x, x, x) = -\frac{11\pi^4}{180} - 9(\text{Ls}_2)^2 + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.55)$$

$$H(0, 0, x, 0, x, x) = -\frac{\pi^4}{10} + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.56)$$

$$H(0, 0, x, x, 0, x) = -\frac{\pi^4}{24} - \frac{27}{2}(\text{Ls}_2)^2 + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.57)$$

$$H(0, 0, x, x, x, x) = -\frac{77\pi^4}{1080} - \frac{27}{2}(\text{Ls}_2)^2 + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.58)$$

$$H(0, x, x, x, 0, x) = 32\text{Li}_4(1/2) - \frac{11\pi^4}{45} + \frac{4}{3}\ln^2(2)[\ln^2(2) - \pi^2] + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.59)$$

$$H(0, x, x, x, x, x) = \frac{7\pi^4}{30} - 6(\text{Ls}_2)^2 + 4\pi\text{Ls}_3 - 6\text{Ls}'_4 - \frac{26}{3}\ln(3)\zeta_3 + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.60)$$

$$H(x, x, x, x, x, x) = 16\text{Li}_4(1/2) - \frac{17\pi^4}{90} + \frac{2}{3}\ln^2(2)[\ln^2(2) - \pi^2] - 9(\text{Ls}_2)^2 + 6\zeta_3[1 - \overline{\ln}(x)], \quad (5.61)$$

and others obtained by permutations implied by the symmetries of the graphs.

Some cases involving two distinct non-zero masses can also be given analytically. Equation (4.19) of ref. [14] gives $\mathbf{H}(0, 0, x, y, x, x)$ through order ϵ^0 , in terms of Nielsen generalized polylogarithm functions, and eq. (3.27) of ref. [15] provides the ϵ expansion of $\mathbf{G}(y, x, x, x, x)$ in terms of log-sine integrals. For brevity, those results are omitted here. Reference [17] obtained the equivalent of $\mathbf{E}(x, x, y, y)$ and $\mathbf{F}(x, x, y, y)$ to all orders in ϵ in terms of hypergeometric functions. Reference [18] obtained the equivalent of $E(x, x, y, y)$ and $F(x, x, y, y)$. Reference [19] contains the expansions of $\mathbf{G}(x, 0, 0, 0, y)$ and $\mathbf{G}(x, 0, 0, y, y)$ and $\mathbf{E}(0, x, x, y)$ and $\mathbf{F}(x, 0, x, y)$, in eqs. (64), (81), (90), and (90) respectively, while ref. ([20]) contains an expression for $\mathbf{G}(x, 0, 0, 0, y)$ to all orders in ϵ in

terms of hypergeometric functions. Reference [16] found results for $\mathbf{E}(0, 0, x, y)$ and $\mathbf{F}(0, 0, x, y)$ and $\mathbf{F}(x, 0, 0, y)$ to all orders in ϵ in terms of hypergeometric functions. Reference [21] also found results for the ϵ expansions of the equivalents of $\mathbf{E}(0, 0, x, y)$ and $\mathbf{F}(x, 0, 0, y)$ in terms of harmonic polylogarithms. Each of those results can be written in terms of only ordinary polylogarithms up through order ϵ^0 . We have also solved the differential equations to obtain a few more cases involving two distinct non-zero masses. Below we list only the cases that can be written in terms of ordinary polylogarithms. This includes the 4-propagator cases:

$$\begin{aligned} E(0, 0, x, y) = & xy[-2\text{Li}_3(1 - x/y) - 2\text{Li}_3(1 - y/x) + (1/3)\overline{\text{ln}}^3(x) - (1/6)\overline{\text{ln}}^3(y) \\ & - \overline{\text{ln}}^2(x)\overline{\text{ln}}(y) + (1/2)\overline{\text{ln}}(x)\overline{\text{ln}}^2(y) + 2\overline{\text{ln}}(x)\overline{\text{ln}}(y) - 2\overline{\text{ln}}(x) - 2\overline{\text{ln}}(y) + 8\zeta_3/3 \\ & + 11/6] + [(x^2 - y^2)/2 + xy \ln(x/y)]\text{Li}_2(1 - x/y) - (x^2/2)\overline{\text{ln}}(x)\overline{\text{ln}}(y) \\ & + [(x^2 - y^2)/4]\overline{\text{ln}}^2(y) + (13x^2/8)\overline{\text{ln}}(x) + (13y^2/8)\overline{\text{ln}}(y) - 133(x^2 + y^2)/48, \end{aligned} \quad (5.62)$$

$$\begin{aligned} F(x, 0, 0, y) = & 2y\text{Li}_3(1 - x/y) + 2y\text{Li}_3(1 - y/x) + [y \ln(y/x) - x + y]\text{Li}_2(1 - x/y) \\ & - (y/3)\overline{\text{ln}}^3(x) + (y/6)\overline{\text{ln}}^3(y) + y\overline{\text{ln}}^2(x)\overline{\text{ln}}(y) - (y/2)\overline{\text{ln}}(x)\overline{\text{ln}}^2(y) \\ & + (x - 2y)\overline{\text{ln}}(x)\overline{\text{ln}}(y) + [(y - x)/2]\overline{\text{ln}}^2(y) + (5y/2 - 11x/4)\overline{\text{ln}}(x) \\ & - (y/2)\overline{\text{ln}}(y) + 47x/12 + y/6 - 8\zeta_3 y/3, \end{aligned} \quad (5.63)$$

$$\begin{aligned} \overline{F}(0, 0, x, y) = & -2x\text{Li}_3(1 - y/x) - 2y\text{Li}_3(1 - x/y) + [x\overline{\text{ln}}(x) - y\overline{\text{ln}}(y) - x + y]\text{Li}_2(1 - x/y) \\ & + (x/3)\overline{\text{ln}}^3(x) - (y/6)\overline{\text{ln}}^3(y) - x\overline{\text{ln}}^2(x)\overline{\text{ln}}(y) + (x/2)\overline{\text{ln}}(x)\overline{\text{ln}}^2(y) + x\overline{\text{ln}}(x)\overline{\text{ln}}(y) \\ & + [(y - x)/2]\overline{\text{ln}}^2(y) - (x/2)\overline{\text{ln}}(x) - (y/2)\overline{\text{ln}}(y) + (x + y)(1/6 + 4\zeta_3/3), \end{aligned} \quad (5.64)$$

$$\begin{aligned} E(x, x, y, y) = & -2(x - y)^2[\text{Li}_3(1 - x/y) + \text{Li}_3(1 - y/x) + \ln(y/x)\text{Li}_2(1 - x/y) \\ & - (1/6)\overline{\text{ln}}^3(x) + (1/3)\overline{\text{ln}}^3(y) - 7\zeta_3/3] + (2xy - 2x^2 - y^2)\overline{\text{ln}}^2(x)\overline{\text{ln}}(y) \\ & + (2x^2 - 4xy + y^2)\overline{\text{ln}}(x)\overline{\text{ln}}^2(y) + x(3x/2 - y)\overline{\text{ln}}^2(x) + y(3y/2 - x)\overline{\text{ln}}^2(y) \\ & + 10xy\overline{\text{ln}}(x)\overline{\text{ln}}(y) - x(8y + 3x/4)\overline{\text{ln}}(x) - y(8x + 3y/4)\overline{\text{ln}}(y) \\ & - 89(x^2 + y^2)/24 + 22xy/3, \end{aligned} \quad (5.65)$$

$$\begin{aligned} F(x, x, y, y) = & 2(x - y)[\text{Li}_3(1 - x/y) + \text{Li}_3(1 - y/x) + \ln(y/x)\text{Li}_2(1 - x/y) - (1/6)\overline{\text{ln}}^3(x) \\ & + (1/3)\overline{\text{ln}}^3(y) - \overline{\text{ln}}(x)\overline{\text{ln}}^2(y) - 7\zeta_3/3] + (2x - y)\overline{\text{ln}}^2(x)\overline{\text{ln}}(y) - x\overline{\text{ln}}^2(x) \\ & - 4y\overline{\text{ln}}(x)\overline{\text{ln}}(y) + y\overline{\text{ln}}^2(y) + (5y - 3x/4)\overline{\text{ln}}(x) - y\overline{\text{ln}}(y) + 49x/12 + y/3, \end{aligned} \quad (5.66)$$

$$\begin{aligned} E(0, x, y, y) = & y(y - x)\left[4\text{Li}_3(-k) + (1/3)\ln^3(k) - (1/3)\overline{\text{ln}}^3(y) + (\pi^2/3)\ln(k) - 4\zeta_3/3\right] \\ & - (x + 2y)\sqrt{x^2 - 4xy}\left[\text{Li}_2(-k) + (1/4)\ln^2(k) + \pi^2/12\right] + [(x^2 + 6y^2)/4]\overline{\text{ln}}^2(y) \\ & - xy\overline{\text{ln}}(x)\overline{\text{ln}}^2(y) + [x(8y - x)/2]\overline{\text{ln}}(x)\overline{\text{ln}}(y) + [x(13x - 32y)/8]\overline{\text{ln}}(x) \\ & - [y(16x + 3y)/4]\overline{\text{ln}}(y) - 133x^2/48 + 11xy/3 - 89y^2/24, \end{aligned} \quad (5.67)$$

$$\begin{aligned} F(x, 0, y, y) = & 4y\text{Li}_3(-k) + \sqrt{x^2 - 4xy}\left[2\text{Li}_2(-k) + (1/2)\ln^2(k) + \pi^2/6\right] \\ & + (y/3)[\ln^3(k) - \overline{\text{ln}}^3(y) + \pi^2\ln(k) + 1 - 4\zeta_3 + 3\overline{\text{ln}}(x)\overline{\text{ln}}^2(y) - 3\overline{\text{ln}}(y)] \\ & + (y - x/2)\overline{\text{ln}}^2(y) + (x - 4y)\overline{\text{ln}}(x)\overline{\text{ln}}(y) + (5y - 11x/4)\overline{\text{ln}}(x) + 47x/12, \end{aligned} \quad (5.68)$$

$$\begin{aligned} F(y, 0, y, x) = & (x - 2y)\left[2\text{Li}_3(-k) + (1/6)\ln^3(k) - (1/6)\overline{\text{ln}}^3(y) + (\pi^2/6)\ln(k) - 2\zeta_3/3\right] \\ & + \sqrt{x^2 - 4xy}\left[2\text{Li}_2(-k) + (1/2)\ln^2(k) + \pi^2/6\right] + (x/2)\overline{\text{ln}}(x)[\overline{\text{ln}}^2(y) - 1] \end{aligned}$$

$$-x\overline{\ln}(x)\overline{\ln}(y) - (y+x/2)\overline{\ln}^2(y) + (5x/2 - 3y/4)\overline{\ln}(y) + x/6 + 49y/12, \quad (5.69)$$

with, in the last three equations,

$$k \equiv (1 - \sqrt{1 - 4y/x}) / (1 + \sqrt{1 - 4y/x}). \quad (5.70)$$

Equations (5.62) and (5.63) are equivalent to results already found by ref. [21]. Equations (5.65) and (5.66) are equivalent to results obtained by refs. [17] and [18]. Equations (5.67), (5.68) and (5.69) are equivalent to results already found in eq. (90) of ref. [19]. Of course, the corresponding \overline{F} integrals can also be obtained from the results above, using eq. (2.41).

The 5-propagator integrals with two distinct non-zero masses that we have been able to find analytically in terms of ordinary polylogarithms are:

$$\begin{aligned} G(0, 0, 0, x, y) = & 2y\text{Li}_3(1 - x/y) + 2x\text{Li}_3(1 - y/x) + [3(x - y) - x\overline{\ln}(x) + y\overline{\ln}(y)]\text{Li}_2(1 - x/y) \\ & - (x/3)\overline{\ln}^3(x) + (y/6)\overline{\ln}^3(y) + x\overline{\ln}^2(x)\overline{\ln}(y) - (x/2)\overline{\ln}(x)\overline{\ln}^2(y) \\ & - 3x\overline{\ln}(x)\overline{\ln}(y) + [3(x - y)/2]\overline{\ln}^2(y) + (11x/2)\overline{\ln}(x) + (11y/2)\overline{\ln}(y) \\ & - (15/2 + 4\zeta_3/3)(x + y), \end{aligned} \quad (5.71)$$

$$\begin{aligned} G(0, 0, x, 0, y) = & (x + y)[-2\text{Li}_3(1 - x/y) - 2\text{Li}_3(1 - y/x) - (1/6)\overline{\ln}^3(y) + (1/3)\overline{\ln}^3(x) \\ & + 8\zeta_3/3 - \pi^2/6 - 15/2] + [2(x - y) + (x + y)\ln(x/y)]\text{Li}_2(1 - x/y) \\ & + (x/2 + y)\overline{\ln}(x)\overline{\ln}(y)\ln(y/x) - (x/2)\overline{\ln}^2(x) + (x - 3y/2)\overline{\ln}^2(y) \\ & - 2x\overline{\ln}(x)\overline{\ln}(y) + [(33x + \pi^2y)/6]\overline{\ln}(x) + [(33y + \pi^2x)/6]\overline{\ln}(y), \end{aligned} \quad (5.72)$$

$$\begin{aligned} G(x, 0, 0, 0, y) = & (y - x)\{\text{Li}_3(1 - x/y) + \text{Li}_3(1 - y/x) + [\overline{\ln}(y) - 2]\text{Li}_2(1 - x/y) \\ & - (1/6)\overline{\ln}^3(x) + (1/3)\overline{\ln}^3(y) - (1/2)\overline{\ln}(x)\overline{\ln}^2(y) - (\pi^2/6)\overline{\ln}(x) - \zeta_3/3\} \\ & + (y/2)\overline{\ln}^2(x)\overline{\ln}(y) - x\overline{\ln}^2(x) - 2x\overline{\ln}(x)\overline{\ln}(y) + (x - 3y/2)\overline{\ln}^2(y) \\ & + 4x\overline{\ln}(x) + (33/6 + \pi^2/6)y\overline{\ln}(y) - (7 + \pi^2)x/3 - (15/2 + \pi^2/6)y, \end{aligned} \quad (5.73)$$

$$\begin{aligned} G(x, 0, x, 0, y) = & -2y\text{Li}_3(1 - x/y) - 2y\text{Li}_3(1 - y/x) + [3x - 3y + (2y - x)\overline{\ln}(x) \\ & - y\overline{\ln}(y)]\text{Li}_2(1 - x/y) + (y/3)\overline{\ln}^3(x) - (y/6)\overline{\ln}^3(y) + (x - y)\overline{\ln}^2(x)\overline{\ln}(y) \\ & + (y - x/2)\overline{\ln}(x)\overline{\ln}^2(y) - 2x\overline{\ln}^2(x) - 3x\overline{\ln}(x)\overline{\ln}(y) + [3(x - y)/2]\overline{\ln}^2(y) \\ & + (19x/2)\overline{\ln}(x) + (11y/2)\overline{\ln}(y) - 59x/6 - 15y/2 + 8\zeta_3y/3, \end{aligned} \quad (5.74)$$

$$\begin{aligned} G(x, 0, y, 0, y) = & -[2(x - y)^2/x]\text{Li}_3(1 - x/y) + 2(x - y)[2 - \overline{\ln}(y)]\text{Li}_2(1 - x/y) \\ & + [(y - 2x)/3]\overline{\ln}^3(y) + [x\overline{\ln}(x) + 2x - 3y]\overline{\ln}^2(y) + 4x\overline{\ln}(x)[1 - \overline{\ln}(y)] \\ & + 11y\overline{\ln}(y) - 7x/3 - 15y + 2(1 + y/x)(3y - x)\zeta_3/3, \end{aligned} \quad (5.75)$$

$$\begin{aligned} G(0, x, x, y, y) = & (\sqrt{x} - \sqrt{y})^2 \left[2\text{Li}_3(1 - x/y) + 2\text{Li}_3(1 - y/x) + 2\ln(y/x)\text{Li}_2(1 - x/y) \right. \\ & \left. - (1/3)\overline{\ln}^3(x) \right] + \sqrt{xy} \left[-32\text{Li}_3(-\sqrt{x/y}) + 16\ln(x/y)\text{Li}_2(-\sqrt{x/y}) \right. \\ & \left. + 4\ln^2(x/y)\ln(1 + \sqrt{x/y}) + 2\overline{\ln}(x)\overline{\ln}(y)\ln(y/x) - (2/3)\overline{\ln}^3(y) + 4\zeta_3 \right] \\ & + (x + y)[(2/3)\overline{\ln}^3(y) - 15 - 14\zeta_3/3] + (2x + y)\overline{\ln}(x)\overline{\ln}(y)\ln(x/y) \\ & - 3x\overline{\ln}^2(x) - 3y\overline{\ln}^2(y) + 11x\overline{\ln}(x) + 11y\overline{\ln}(y), \end{aligned} \quad (5.76)$$

$$G(x, 0, x, y, y) = -F(x, 0, y, y) + [2 - \overline{\ln}(x)]I(x, y, y) + x\overline{\ln}(x)/4 + 2y\overline{\ln}(y) - 11x/12 - 14y/3, \quad (5.77)$$

$$G(x, 0, x, x, y) = -F(x, 0, x, y) + [2 - \overline{\ln}(x)]I(x, x, y) + 5x\overline{\ln}(x)/4 + y\overline{\ln}(y) - 13x/4 - 7y/3, \quad (5.78)$$

$$G(y, x, x, x, x) = (4 - y/x)\overline{F}(y, 0, x, x) + [8 - y/x - 2\overline{\ln}(x)]I(x, x, y) + y[-8/3 - 2\overline{\ln}(x) - \overline{\ln}(y) + 4\overline{\ln}(x)\overline{\ln}(y) - \overline{\ln}^2(x)\overline{\ln}(y)] + x[26/3 - 16\overline{\ln}(x) + 6\overline{\ln}^2(x)] + (y^2/4x)[17/3 - 3\overline{\ln}(y)] + (8x - 2y)\zeta_3, \quad (5.79)$$

$$G(x, x, y, x, y) = (y/x - 1)F(x, x, y, y) + y(y - 4x)F(y, 0, x, x)/2x^2 + [8x^2 - 4xy + y^2 + 2x(y - 2x)\overline{\ln}(x) - 2xy\overline{\ln}(y)]I(x, x, y)/2x^2 + x[27/4 + (47/4)\overline{\ln}(x) + 6\overline{\ln}^2(x) - \overline{\ln}^3(x)] + y[-217/12 + (47/4)\overline{\ln}(x) + 7\overline{\ln}(y) - 3\overline{\ln}^2(x) - 4\overline{\ln}(x)\overline{\ln}(y) + \overline{\ln}^2(x)\overline{\ln}(y)] + (y^2/x)[29/6 - 4\overline{\ln}(x) - (11/2)\overline{\ln}(y) + 4\overline{\ln}(x)\overline{\ln}(y) + \overline{\ln}^2(y) - \overline{\ln}(x)\overline{\ln}^2(y)] + (y^3/8x^2)[-17/3 + 3\overline{\ln}(y)] + (8y - 2y^2/x)\zeta_3, \quad (5.80)$$

$$G(x, 0, 0, y, y) = \sqrt{x^2 - 4xy} \left\{ 2\text{Li}_3(-k) + 4\text{Li}_3(k/[1 + k]) + \text{Li}_2(-k)[2\overline{\ln}(y) - 4] + (1/12)\ln^3(k) + \ln^2(k)[\overline{\ln}(x) + \overline{\ln}(y) - 4]/4 - \ln(k)[(1/4)\ln^2(x/y) + \pi^2/6] - (1/12)\ln^3(x/y) + \pi^2(\overline{\ln}(x) - 2)/6 - 2\zeta_3 \right\} + (y/3 - x/6)\overline{\ln}^3(y) + (x/2)\overline{\ln}^2(x)\overline{\ln}(y) + (x - 3y)\overline{\ln}^2(y) - 2x\overline{\ln}(x)\overline{\ln}(y) - x\overline{\ln}^2(x) + 4x\overline{\ln}(x) + [\pi^2 x/6 + 11y]\overline{\ln}(y) + 4(x - 2y)\zeta_3/3 - (7 + \pi^2)x/3 - 15y, \quad (5.81)$$

where k in the last equation was given in eq. (5.70). Equations (5.73) and (5.81) are equivalent to the results already obtained in eqs. (64) and (81) of ref. [19]. The equivalent of eq. (5.75) has also been obtained in terms of harmonic polylogarithms in ref. [21].

In addition to the analytical cases, we find various identities that can be obtained by requiring the absence of pole singularities in the derivatives of the basis integrals for special values of the input squared masses. For example, the following identities allow for all remaining cases of G with first argument vanishing to be written in terms of integral functions with fewer propagators:

$$G(0, u, v, y, z) = \left\{ vF(v, u, y, z) - uF(u, v, y, z) + [A(v) - A(u)]I(0, y, z) + [uA(u) - vA(v)]/4 \right\} / (u - v) + \left\{ zF(z, y, u, v) - yF(y, z, u, v) + [A(z) - A(y)]I(0, u, v) + [yA(y) - zA(z)]/4 \right\} / (y - z) - 2(u + v + y + z)/3, \quad (5.82)$$

$$G(0, u, u, y, z) = 2 \left\{ zF(z, y, u, u) - yF(y, z, u, u) + [A(z) - A(y)]I(0, u, u) + [yA(y) - zA(z)]/4 \right\} / (y - z) - F(u, u, y, z) - \overline{\ln}(u)I(0, y, z) - A(y) - A(z) + A(u)/4, \quad (5.83)$$

This is useful because we find that when the first argument of G vanishes, it tends to be especially

sensitive to non-negligible numerical error from the Runge-Kutta integration described in sections VI and VII below, but we can always replace that value by the results of one of eqs. (5.71), (5.72), (5.76), (5.82), or (5.83).

Another special identity is:

$$\begin{aligned}
G(x, u, v, y, z) \Big|_{u=(\sqrt{x}-\sqrt{v})^2} &= (r-1) F(u, v, y, z) - r F(v, u, y, z) \\
&+ [1 - r A(v)/v + (r-1) A(u)/u] I(x, y, z) \\
&+ (r/4)[A(v) + v] + [(1-r)/4][A(u) + u] + A(y) + A(z) \\
&+ (19x - 41u - 41v - 32y - 32z)/24,
\end{aligned} \tag{5.84}$$

where $r = \sqrt{v/x}$. In the special case $u = 0$, this reduces to

$$\begin{aligned}
G(x, 0, x, y, z) &= -F(x, 0, y, z) + [1 - A(x)/x] I(x, y, z) + A(x)/4 + A(y) + A(z) \\
&- (2x + 4y + 4z)/3,
\end{aligned} \tag{5.85}$$

which in turn has the fully analytic (in terms of ordinary polylogarithms) special cases of eqs. (5.74), (5.77), and (5.78). Also, the following 4-propagator integral identity provides a useful check when one of the squared mass arguments vanishes:

$$\begin{aligned}
0 &= u(u-y-z)F(u, 0, y, z) + y(y-u-z)F(y, 0, u, z) + z(z-u-y)F(z, 0, u, y) \\
&+ \lambda(u, y, z)I(u, y, z) + 2A(u)A(y)A(z) - 2uA(y)A(z) - 2yA(u)A(z) - 2zA(u)A(y) \\
&+ (3u - 4y - 4z)(u-y-z)A(u)/4 + (3y - 4u - 4z)(y-u-z)A(y)/4 \\
&+ (3z - 4u - 4y)(z-u-y)A(z)/4 \\
&+ 2(u^2y + u^2z + y^2u + y^2z + z^2u + z^2y - u^3 - y^3 - z^3)/3.
\end{aligned} \tag{5.86}$$

These identities can be useful in reducing analytical expressions before numerical evaluation.

Finally, in all cases with two squared mass scales x, y , it is possible to find series expansions in expansion parameters such as y/x , x/y , $(1-y/x)$, and $(1-y/4x)$, with overlapping regions of convergence that in their union cover all x, y . In the code **3VIL** described below, we have incorporated such series results for the cases $H(0, 0, 0, 0, x, y)$, $H(0, 0, x, y, 0, 0)$, $H(x, x, 0, 0, y, y)$, $H(x, x, 0, y, x, x)$, $H(0, 0, x, 0, y, y)$, $H(0, 0, x, y, x, x)$, $H(0, 0, x, x, x, y)$, $H(0, x, y, x, x, y)$, and $H(0, 0, x, y, x, y)$, together with the only subordinate 4-propagator and 5-propagator integrals of these that are not already given above analytically, namely $\overline{F}(0, x, x, y)$, $G(x, 0, 0, x, y)$, $G(y, 0, x, x, x)$, $G(x, 0, y, x, y)$. The coefficients of the terms in the series expansions are implemented as pre-computed numerical values. In future updates of **3VIL**, we plan to include more such special cases.

VI. DIFFERENTIAL EQUATIONS FOR NUMERICAL EVALUATION

In this section we describe the differential equations method used for finding the 3-loop basis integrals in the case of generic squared mass arguments. The equations described below are implemented in the software package **3VIL**, as described in the following section.

For a given master tetrahedral topology corresponding to a basis integral

$$H(u, v, w, x, y, z), \quad (6.1)$$

the list of subordinate 3-loop basis integrals G obtained by removing one propagator is:

$$G(w, u, z, v, y), \quad G(x, u, v, y, z), \quad G(u, v, x, w, z), \quad (6.2)$$

$$G(y, v, w, x, z), \quad G(v, u, x, w, y), \quad G(z, u, w, x, y). \quad (6.3)$$

The list of subordinate \overline{F} integrals obtained by removing a second propagator is

$$\overline{F}(w, u, x, y), \quad \overline{F}(w, v, x, z), \quad \overline{F}(x, u, w, y), \quad \overline{F}(x, v, w, z), \quad (6.4)$$

$$\overline{F}(u, v, y, z), \quad \overline{F}(u, w, x, y), \quad \overline{F}(y, u, v, z), \quad \overline{F}(y, u, w, x), \quad (6.5)$$

$$\overline{F}(v, u, y, z), \quad \overline{F}(v, w, x, z), \quad \overline{F}(z, u, v, y), \quad \overline{F}(z, v, w, x). \quad (6.6)$$

Also, there are associated 2-loop basis integrals, obtained by removing from $H(u, v, w, x, y, z)$ any three propagators forming a complete loop:

$$I(u, v, x), \quad I(x, y, z), \quad I(u, x, y), \quad I(v, x, z), \quad I(u, w, z), \quad I(v, w, y), \quad (6.7)$$

$$I(u, w, y), \quad I(v, w, z), \quad I(v, y, z), \quad I(w, x, y), \quad I(u, v, z), \quad I(u, w, x), \quad (6.8)$$

$$I(u, y, z), \quad I(w, x, z), \quad I(u, v, y), \quad I(v, w, x). \quad (6.9)$$

Although the I functions are known analytically in terms of dilogarithms, in practice it is more efficient to treat them as dependent variables and solve for them simultaneously with the 3-loop basis functions.

We now introduce a dimensionless independent variable t , and an arbitrary[†] reference squared mass a , and define the quantities

$$\begin{aligned} U &= a + t(u - a), & V &= a + t(v - a), & W &= a + t(w - a), \\ X &= a + t(x - a), & Y &= a + t(y - a), & Z &= a + t(z - a). \end{aligned} \quad (6.10)$$

Now consider the 3-loop and 2-loop basis integrals, generically denoted f_i , as functions of arguments (U, V, W, X, Y, Z) , or equivalently as functions of u, v, w, x, y, z and t . These functions satisfy coupled first-order differential equations of the general form:

$$\frac{df_i}{dt} = \sum_j c_{ij} f_j + c_i. \quad (6.11)$$

[†] In principle, the results should not depend on the choice of a . By default, 3VIL chooses $a = 2 |\text{Max}(u, v, w, x, y, z)|$, which we find avoids some numerical complications, with some exceptions noted below which require a different choice. As an option, a can be specified at run time. Changing a allows a check on the numerical errors.

Here, the c_{ij} are ratios of polynomials in the squared masses and t and, in the case where f_j is an I function, also linear functions of the logarithms $\overline{\ln}(U), \overline{\ln}(V)$, etc. The c_i are up to cubic functions of the logarithms when i is a 3-loop integral, and quadratic functions of the logarithms when f_i is an I integral. The differential equations are given explicitly below. These coupled differential equations in t can be solved numerically by Runge-Kutta, using appropriate boundary conditions. At $t = 0$, all of the propagator squared masses are equal to a , while at the endpoint of the integration $t = 1$ we have $(U, V, W, X, Y, Z) = (u, v, w, x, y, z)$ equal to the desired values.

We now provide the derivatives of the basis integrals with respect to t . It is convenient to first define some auxiliary functions, in addition to the functions λ and ψ defined in eqs. (4.3) and (4.8) respectively:

$$\kappa(x, y, z) = x^2 + y^2 + z^2 - xy - xz - yz, \quad (6.12)$$

$$\Delta(w, x, y, z) = \lambda(x, y, z) + 2w(x + y + z) - 3w^2, \quad (6.13)$$

$$\phi(w, x, y, z) = \psi(w, x, y, z) + 8a(w + x - y - z)(w - x + y - z)(w - x - y + z), \quad (6.14)$$

Then define:

$$r_{\pm}(x, y, z) = a[x + y + z - 3a \pm 2\sqrt{\kappa(x, y, z)}] / \lambda(a - x, a - y, a - z), \quad (6.15)$$

and, if $\Delta(w, x, y, z) \neq 0$,

$$r_4(w, x, y, z) = 8a(w + x - y - z)(w - x + y - z)(w - x - y + z) / \phi(w, x, y, z), \quad (6.16)$$

while in the alternative,

$$r_4(w, x, y, z) = a / (a - w) \quad [\text{if } \Delta(w, x, y, z) = 0]. \quad (6.17)$$

Note that if $\Delta(w, x, y, z) \neq 0$, one must be careful not to choose a to be the specific value such that $\phi(w, x, y, z) = 0$; otherwise a singularity would occur in eq. (6.16). These are the exceptions referred to in the previous footnote. Our program **3VIL** automatically ensures that a is chosen appropriately.

Then we can write:

$$\begin{aligned} \frac{d}{dt} I(X, Y, Z) = & c_{II}(x, y, z) I(X, Y, Z) + c_{ILL}(x, y, z) \overline{\ln}(X) \overline{\ln}(Y) \\ & + c_{ILL}(x, z, y) \overline{\ln}(X) \overline{\ln}(Z) + c_{ILL}(y, z, x) \overline{\ln}(Y) \overline{\ln}(Z) \\ & + c_{IL}(x, y, z) \overline{\ln}(X) + c_{IL}(y, x, z) \overline{\ln}(Y) + c_{IL}(z, x, y) \overline{\ln}(Z) + c_I(x, y, z), \end{aligned} \quad (6.18)$$

where we suppress the a and t dependences when writing the arguments of the coefficient functions. These are given by

$$c_{II}(x, y, z) = \frac{1}{2(t - p_+)} + \frac{1}{2(t - p_-)}, \quad (6.19)$$

$$c_{ILL}(x, y, z) = \frac{c_{ILL+}}{t - p_+} + \frac{c_{ILL-}}{t - p_-}, \quad (6.20)$$

$$c_{IL}(x, y, z) = a - x + \frac{c_{IL+}}{t - p_+} + \frac{c_{IL-}}{t - p_-}, \quad (6.21)$$

$$c_I(x, y, z) = 2x + 2y + 2z - 6a + \frac{c_{I+}}{t - p_+} + \frac{c_{I-}}{t - p_-}, \quad (6.22)$$

with simple poles at

$$p_{\pm} = r_{\pm}(x, y, z), \quad (6.23)$$

and coefficients:

$$c_{ILL\pm} = [a + (x + y - z - a)p_{\pm}]/4, \quad (6.24)$$

$$c_{IL\pm} = (a - x)p_{\pm} - a, \quad (6.25)$$

$$c_{I\pm} = 5[3a + (x + y + z - 3a)p_{\pm}]/4. \quad (6.26)$$

In 3VIL, the t -independent coefficients appearing in eqs. (6.18)-(6.26) and similar equations below are computed only once, before the Runge-Kutta running begins.

Similarly, we find:

$$\begin{aligned} \frac{d}{dt}\overline{F}(U, V, Y, Z) = & c_{FF1}(u, v, y, z)F(U, V, Y, Z) + c_{FF2}(u, v, y, z)F(V, U, Y, Z) \\ & + c_{FF2}(u, y, v, z)F(Y, U, V, Z) + c_{FF2}(u, z, v, y)F(Z, U, V, Y) \\ & + c_{FLLL1}(u, v, y, z)\overline{\ln}(V)\overline{\ln}(Y)\overline{\ln}(Z) + c_{FLLL2}(u, v, y, z)\overline{\ln}(U)\overline{\ln}(V)\overline{\ln}(Y) \\ & + c_{FLLL2}(u, v, z, y)\overline{\ln}(U)\overline{\ln}(V)\overline{\ln}(Z) + c_{FLLL2}(u, y, z, v)\overline{\ln}(U)\overline{\ln}(Y)\overline{\ln}(Z) \\ & + c_{FLL1}(u, v, y, z)\overline{\ln}(V)\overline{\ln}(Y) + c_{FLL1}(u, v, z, y)\overline{\ln}(V)\overline{\ln}(Z) \\ & + c_{FLL1}(u, y, z, v)\overline{\ln}(Y)\overline{\ln}(Z) + c_{FLL2}(u, v, y, z)\overline{\ln}(U)\overline{\ln}(V) \\ & + c_{FLL2}(u, y, v, z)\overline{\ln}(U)\overline{\ln}(Y) + c_{FLL2}(u, z, v, y)\overline{\ln}(U)\overline{\ln}(Z) \\ & + c_{FIL}(u, v, y, z)\overline{\ln}(U)I(V, Y, Z) + c_{FI}(u, v, y, z)I(V, Y, Z) \\ & + c_{FL1}(u, v, y, z)\overline{\ln}(U) + c_{FL2}(u, v, y, z)\overline{\ln}(V) + c_{FL2}(u, y, v, z)\overline{\ln}(Y) \\ & + c_{FL2}(u, z, v, y)\overline{\ln}(Z) + c_F(u, v, y, z), \end{aligned} \quad (6.27)$$

Note that the right side contains F functions, which in the 3VIL code are expressed in terms of \overline{F} functions using eq. (2.41). The coefficient functions on the right side again can be written as sums over simple poles in t :

$$c_{FF1}(u, v, y, z) = \frac{3}{4t} + \frac{1}{4(t - p_3)}, \quad (6.28)$$

$$c_{FF2}(u, v, y, z) = -\frac{1}{4t} + \frac{c_{FF22}}{t - p_2} + \frac{c_{FF23}}{t - p_3}, \quad (6.29)$$

$$c_{FLLL1}(u, v, y, z) = \frac{3a}{4t} + \frac{c_{FLLL12}}{t - p_2} + \frac{c_{FLLL13}}{t - p_3}, \quad (6.30)$$

$$c_{FLLL2}(u, v, y, z) = -\frac{a}{4t} + \frac{c_{FLLL23}}{t-p_3} + \frac{c_{FLLL24}}{t-p_4} + \frac{c_{FLLL25}}{t-p_5}, \quad (6.31)$$

$$c_{FLL1}(u, v, y, z) = -\frac{a}{t} + \frac{c_{FLL12}}{t-p_2} + \frac{c_{FLL13}}{t-p_3}, \quad (6.32)$$

$$c_{FLL2}(u, v, y, z) = \frac{a}{t} + \frac{c_{FLL23}}{t-p_3} + \frac{c_{FLL24}}{t-p_4} + \frac{c_{FLL25}}{t-p_5}, \quad (6.33)$$

$$c_{FIL}(u, v, y, z) = \frac{1}{2(t-p_4)} + \frac{1}{2(t-p_5)}, \quad (6.34)$$

$$c_{FI}(u, v, y, z) = \frac{1}{t-p_2}, \quad (6.35)$$

$$c_{FL1}(u, v, y, z) = u - a - \frac{63a}{16t} + \frac{c_{FL13}}{t-p_3} + \frac{c_{FL14}}{t-p_4} + \frac{c_{FL15}}{t-p_5}, \quad (6.36)$$

$$c_{FL2}(u, v, y, z) = a - v + \frac{21a}{16t} + \frac{c_{FL22}}{t-p_2} + \frac{c_{FL23}}{t-p_3}, \quad (6.37)$$

$$c_F(u, v, y, z) = -\frac{13}{4}a - \frac{11}{4}u + 2v + 2y + 2z + \frac{c_{F2}}{t-p_2} + \frac{c_{F3}}{t-p_3}, \quad (6.38)$$

where the coefficients c_{FF22} etc. on the right side are independent of t , and there are simple poles in t at

$$p_1 = 0, \quad (6.39)$$

$$p_2 = a/(a-u), \quad (6.40)$$

$$p_3 = r_4(u, v, y, z), \quad (6.41)$$

$$p_{4,5} = r_{\pm}(v, y, z). \quad (6.42)$$

Note that there are always poles at $t = 0$. If a squared mass argument vanishes, then there will also be a pole at $t = 1$. The explicit forms for some of the t -independent coefficients on the right sides of eqs. (6.28)-(6.38) are somewhat complicated, so they are relegated to an ancillary electronic file called `dFbardtcoeffs.txt`, which is included with the arXiv submission for this paper. There are two separate forms for these coefficients, depending on whether $\Delta(u, v, y, z)$ is zero or non-zero.

The differential equations for the G functions have the form:

$$\begin{aligned} \frac{d}{dt}G(W, U, Z, V, Y) = & c_{GG}(w, u, z, v, y)G(W, U, Z, V, Y) \\ & + c_{GF}(w, u, z, v, y)[F(U, V, Y, Z) + \overline{\ln}(U)I(V, W, Y)] \\ & + c_{GF}(w, z, u, v, y)[F(Z, U, V, Y) + \overline{\ln}(Z)I(V, W, Y)] \\ & + c_{GF}(w, v, y, u, z)[F(V, U, Y, Z) + \overline{\ln}(V)I(U, W, Z)] \\ & + c_{GF}(w, y, v, u, z)[F(Y, U, V, Z) + \overline{\ln}(Y)I(U, W, Z)] \\ & + c_{GI}(w, u, z, v, y)I(U, W, Z) + c_{GI}(w, v, y, u, z)I(V, W, Y) \\ & + c_{GL}(w, u, z, v, y)U\overline{\ln}(U) + c_{GL}(w, z, u, v, y)Z\overline{\ln}(Z) \\ & + c_{GL}(w, v, y, u, z)V\overline{\ln}(V) + c_{GL}(w, y, v, u, z)Y\overline{\ln}(Y) \\ & + c_G(w, u, z, v, y) \end{aligned} \quad (6.43)$$

where again the F functions on the right side are re-expressed in terms of \overline{F} functions in the 3VIL code using eq. (2.41). The coefficient functions are:

$$c_{GG}(w, u, z, v, y) = -\frac{1}{t-p_1} + \frac{1}{2} \left[\frac{1}{t-p_2} + \frac{1}{t-p_3} + \frac{1}{t-p_4} + \frac{1}{t-p_5} \right], \quad (6.44)$$

$$c_{GF}(w, u, z, v, y) = \frac{c_{GF1}}{t-p_1} + \frac{c_{GF4}}{t-p_4} + \frac{c_{GF5}}{t-p_5}, \quad (6.45)$$

$$c_{GI}(w, u, z, v, y) = \frac{1}{t-p_1} - \frac{1}{t-p_2} - \frac{1}{t-p_3}, \quad (6.46)$$

$$c_{GL}(w, u, z, v, y) = -\frac{1}{2(t-p_2)} - \frac{1}{2(t-p_3)} - \frac{1}{4}c_{GF}(w, u, z, v, y), \quad (6.47)$$

$$c_G(w, u, z, v, y) = c_{G0} + \frac{c_{G1}}{t-p_1} + \frac{c_{G2}}{t-p_2} + \frac{c_{G3}}{t-p_3} + \frac{c_{G4}}{t-p_4} + \frac{c_{G5}}{t-p_5}, \quad (6.48)$$

with simple poles at

$$p_1 = a/(a-w), \quad (6.49)$$

$$p_{2,3} = r_{\pm}(y, v, w), \quad (6.50)$$

$$p_{4,5} = r_{\pm}(u, z, w). \quad (6.51)$$

The coefficients in eq. (6.48) are given by

$$c_{G0} = -11a - w + 3u + 3v + 3y + 3z, \quad (6.52)$$

$$c_{G1} = 11a(4w - u - v - y - z)/12(a-w), \quad (6.53)$$

$$c_{G2,3} = 175a/48 + p_{2,3}(-175a - 19w + 56u + 56z + 41v + 41y)/48, \quad (6.54)$$

$$c_{G4,5} = 175a/48 + p_{4,5}(-175a - 19w + 56v + 56y + 41u + 41z)/48, \quad (6.55)$$

and those in eq. (6.45) are given by, if $u \neq z$:

$$c_{GF1} = (w-u)/(u-z), \quad (6.56)$$

$$c_{GF4,5} = [u-w \pm \sqrt{\kappa(u, w, z)}]/2(u-z), \quad (6.57)$$

while if $u = z$ they are:

$$c_{GF1} = 0, \quad (6.58)$$

$$c_{GF4,5} = \pm \text{sign}(w-u)/4, \quad (6.59)$$

with $\text{sign}(x) = x/|x|$ if $x \neq 0$, and $\text{sign}(0) \equiv 0$.

Finally, the differential equation for H is:

$$\begin{aligned} \frac{d}{dt}H(U, V, W, X, Y, Z) = & c_{HG}(u, v, w, x, y, z) \left\{ G(X, U, V, Y, Z) - F(X, V, W, Z) \right. \\ & \left. - F(X, U, W, Y) + [2 - \overline{\ln}(X)][I(U, W, Z) + I(V, W, Y)] \right\} \end{aligned}$$

$$\begin{aligned}
& +W\overline{\ln}(W) + X\overline{\ln}(X)/2 - 5X/2 - 7W/3 \Big\} + (5 \text{ permutations}) \\
& +c_H(u, v, w, x, y, z),
\end{aligned} \tag{6.60}$$

where the “(5 permutations)” of squared masses (u, v, w, x, y, z) are determined by the tetrahedral symmetry of Figure 2.3, and are given by (u, z, x, w, y, v) and (u, w, v, z, y, x) and (u, x, z, v, y, w) and (w, v, u, y, x, z) and (x, v, y, u, w, z) . The coefficient functions have the forms:

$$c_{HG}(u, v, w, x, y, z) = \frac{c_{HGn}(u, v, w, x, y, z)}{c_{Hd1}(u, v, w, x, y, z)c_{Hd2}(u, v, x)c_{Hd2}(x, y, z)}, \tag{6.61}$$

$$c_H(u, v, w, x, y, z) = \frac{c_{Hn}(u, v, w, x, y, z)}{c_{Hd1}(u, v, w, x, y, z)}\zeta_3 \tag{6.62}$$

where c_{HGn} , c_{Hd1} , c_{Hd2} , and c_{Hn} are polynomials in t of orders 4, 3, 2, and 2, respectively. The roots of the quadratic polynomials $c_{Hd2}(u, v, x)$ and $c_{Hd2}(x, y, z)$ are respectively $t = r_{\pm}(u, v, x)$ and $t = r_{\pm}(x, y, z)$. The cubic polynomial in t appearing in these denominators is:

$$\begin{aligned}
c_{Hd1}(u, v, w, x, y, z) = & -2a^3 + (6a - u - v - w - x - y - z)a^2t + [-6a^2 + 2a(u + v \\
& + w + x + y + z) + u^2 + v^2 + w^2 + x^2 + y^2 + z^2 - uv - uw - vw - ux \\
& - vx - vy - wy - xy - uz - wz - xz - yz]at^2 + [uvx - uwx - vwx - uvy \\
& - uwy + vwy - uxy - wxy - uvz + uwz - vwz - vxz - wxz - uyz - vyz \\
& + xyz + u^2y + uy^2 + v^2z + vz^2 + w^2x + wx^2 + a(uv + uw + vw + ux + vx \\
& + vy + wy + xy + uz + wz + xz + yz - u^2 - v^2 - w^2 - x^2 - y^2 - z^2) \\
& - a^2(u + v + w + x + y + z) + 2a^3]t^3.
\end{aligned} \tag{6.63}$$

In terms of the roots $R_{1,2,3}$ of this cubic polynomial in t , the expression for c_H can be rewritten in a very simple form:

$$c_H(u, v, w, x, y, z) = -2\zeta_3 \left[\frac{1}{t - R_1} + \frac{1}{t - R_2} + \frac{1}{t - R_3} \right]. \tag{6.64}$$

Unfortunately, however, attempting to write c_{HG} as a sum of simple poles leads to extremely complicated expressions related to the solutions of a cubic equations, with residue coefficients that are also singular in a variety of special cases for the squared masses. Therefore, we instead write:

$$c_{HG}(u, v, w, x, y, z) = \left(\sum_{k=0}^4 c_{HGn}^{(k)} t^k \right) / \left(\sum_{k=0}^7 c_{HGd}^{(k)} t^k \right), \tag{6.65}$$

with coefficients $c_{HGn}^{(k)}$ and $c_{HGd}^{(k)}$ that are complicated polynomials in u, v, w, x, y, z . They are given in an ancillary file called `cHG.txt`, both in the generic case and in all special cases involving degenerate masses in which simplification occurs because the numerator and denominator can be reduced by a common factor. This is important for the Runge-Kutta evaluation because it avoids spurious higher-order poles at (or near) $t = 1$ when one or more squared masses vanishes (or is

relatively small). All of the natural special cases involving one or more degenerate squared masses are identified and treated separately within `CHG.txt`. The computer library `3VIL` automatically identifies and deals with these special cases. It should be noted that there are other special cases of squared mass arguments in which our expression for c_{HG} has higher-order poles in t , but those are all unnatural in the sense that they require relationships between squared masses that are not degeneracies and not consequences of any possible symmetry of a quantum field theory. At, and near, such unnatural special points one should be aware that there may be some loss of numerical precision.

From the above results, we can now make a list of all of the poles in t in the complete set of coupled differential equations. They consist of the union of the points: 0, and $a/(a - x_i)$ for $x_i = u, v, w, x, y, z$, and $r_{\pm}(x_i, x_j, x_k)$ for every triplet of arguments of I functions appearing in eq. (6.9), and $r_4(x_i, x_j, x_k, x_l)$ for every quartet of arguments of \overline{F} functions appearing in eq. (6.6), and the three roots R_1 , R_2 , and R_3 of the cubic equation (6.63).

VII. IMPLEMENTATION IN SOFTWARE: 3VIL

In this section, we describe the software package `3VIL`, available at [43], which takes inputs u, v, w, x, y, z and the renormalization scale Q , and outputs the numerical values of all of the basis integrals listed above using either the analytic expressions from section V, or, when those do not apply, a simultaneous Runge-Kutta computation involving their coupled differential equations in t found in the previous section.

The $t = 0$ values are known from eqs. (5.9), (5.38), (5.51), and (5.61), and so in principle could serve as boundary conditions for the Runge-Kutta integration. However, there is a technical difficulty in that some of the coefficients c_{ij} and c_i have unavoidable poles at $t = 0$, even though the basis integral functions are always well-defined and smooth there. Therefore, we instead choose to integrate starting from a small non-zero value of t . This is done by first analytically solving the coupled differential equations as power series expansions in t :

$$I(X, Y, Z) = I(a, a, a) + \sum_{n \geq 1} t^n I^{(n)}(x, y, z; a), \quad (7.1)$$

$$\overline{F}(U, V, Y, Z) = \overline{F}(a, a, a, a) + \sum_{n \geq 1} t^n \overline{F}^{(n)}(u, v, y, z; a), \quad (7.2)$$

$$G(W, U, Z, V, Y) = G(a, a, a, a, a) + \sum_{n \geq 1} t^n G^{(n)}(w, u, z, v, y; a), \quad (7.3)$$

$$H(U, V, W, X, Y, Z) = H(a, a, a, a, a, a) + \sum_{n \geq 1} t^n H^{(n)}(u, v, w, x, y, z; a). \quad (7.4)$$

The leading order (t^0) terms can be read off immediately from eqs. (5.9), (5.38), (5.51), and (5.61), and the coefficients of t^1 are given by:

$$I^{(1)}(x, y, z; a) = (3a - x - y - z) \left[\frac{1}{2} - \sqrt{3} \text{Ls}_2 - \overline{\text{Ln}}(a) + \frac{1}{2} \overline{\text{Ln}}^2(a) \right], \quad (7.5)$$

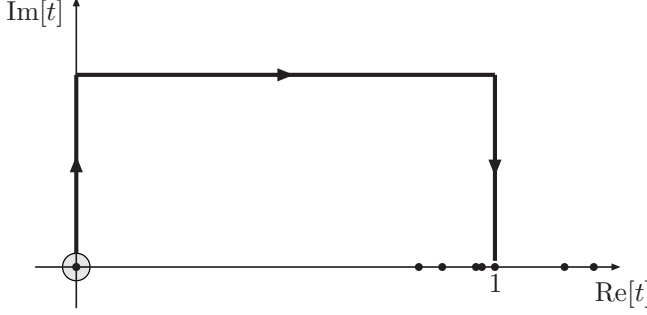


FIG. 7.1: Schematic of the contour in the upper-half complex t plane for simultaneous evaluation of the basis integrals in eqs.(6.1)-(6.9) using Runge-Kutta integration. The integrals are initiated near the origin using an expansion for small $|t| = 0.013$, and the contour finishes at $t = 1$. The dots on the $\text{Re}[t]$ axis represent poles in the coefficients in the differential equations.

$$\begin{aligned} \overline{F}^{(1)}(u, v, y, z; a) = & (7u - 2z - 2y - 2v - a)/6 + 3\sqrt{3}\text{Ls}_2(u - a) + \frac{7}{3}\zeta_3(v + y + z - 3u) \\ & + [\sqrt{3}\text{Ls}_2(v + y + z - 3a) + (a - 7u + 2v + 2y + 2z)/4]\overline{\ln}(a) \\ & + \frac{1}{2}(u - a)\overline{\ln}^2(a) + \frac{1}{6}(3a - v - y - z)\overline{\ln}^3(a), \end{aligned} \quad (7.6)$$

$$\begin{aligned} G^{(1)}(w, u, z, v, y; a) = & 19a/3 + 5w/3 - 2u - 2v - 2y - 2z + 2(\sqrt{3}\text{Ls}_2 + \zeta_3)(u + v + y + z \\ & - w - 3a) + [\sqrt{3}\text{Ls}_2(6a - 2w - u - v - y - z) + 5(u + v + y + z - 4a)/2]\overline{\ln}(a) \\ & + (5a - w - u - v - y - z)\overline{\ln}^2(a) + [(2w + u + v + y + z - 6a)/6]\overline{\ln}^3(a), \end{aligned} \quad (7.7)$$

$$H^{(1)}(u, v, w, x, y, z; a) = \zeta_3(6a - u - v - w - x - y - z)/a. \quad (7.8)$$

We have computed the remaining terms of these expansions up through order t^8 . These results are provided in an ancillary files, called `texpansions.txt`, provided with the arXiv sources for this paper. In 3VIL, we use these expansions to initiate the Runge-Kutta running at a small non-zero value $t = t_{\text{in}}$ with magnitude 0.013, so that the associated numerical relative error is of the order 10^{-16} , comparable to the round-off error for long double arithmetic.

Another complication is that the coefficient functions c_{ij} and c_i also have poles at non-zero t . These poles always lie on the real t axis; their locations were listed at the end of section VI. For many (but not all) choices of inputs u, v, w, x, y, z , one or more of these poles will lie in the range of t between 0 and 1 (for any choice of a). In order to avoid numerical problems when such poles are present with $0 < t < 1$, we promote t to a complex variable, and integrate the coupled differential equations (6.11) in the upper[†] half complex t plane along a contour that avoids the real t axis, as shown in Figure 7.1. By default, the displacement of the contour in the $\text{Im}(t)$ direction is 0.8, but this can be changed by the user at run time. The initial point is chosen to be $t = i|t_{\text{in}}|$ in this case, by default. In the nicer case of inputs u, v, w, x, y, z , and a such that there is no pole in any of the coefficients c_{ij} or c_i for $0 < t < 1$, we save time and numerical accuracy by integrating the coupled differential equation directly along the real axis from $t = |t_{\text{in}}|$ to $t = 1$. The user can also change the default value of the magnitude of the starting point $|t_{\text{in}}|$ from 0.013 to another value at run time.

The Runge-Kutta running is performed with the 6-stage, 5th-order Cash-Karp algorithm [44]

[†] A vacuum loop integral function of real squared mass arguments can have an imaginary part if, and only if, one or more of the arguments is negative. For example, integral functions dependent on one mass scale x are obtained by taking $\overline{\ln}(x) \rightarrow \overline{\ln}(|x|) - i\pi$ for real negative x . More generally, approaching $t = 1$ from above in the complex t plane provides the correct $m^2 - i\epsilon$ Feynman propagator prescription, and thus ensures that the imaginary parts of the integral functions will have the correct signs when one or more squared masses is negative.

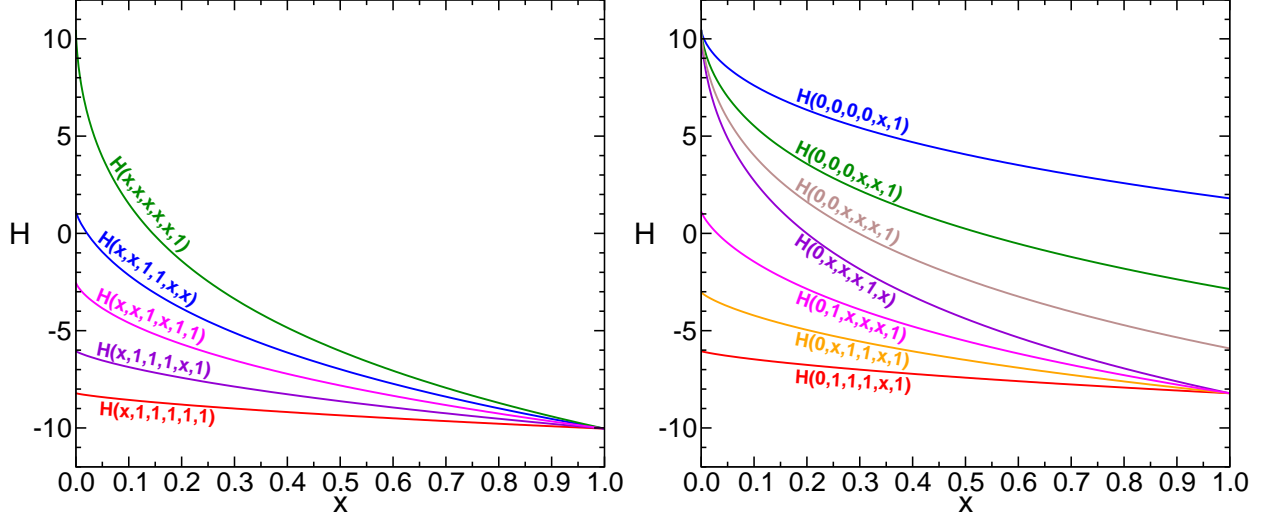


FIG. 7.2: Numerical values of the integral H for selected squared mass arguments, from top to bottom $H(x, x, x, x, 1)$, $H(x, x, 1, 1, x)$, $H(x, x, 1, x, 1)$, $H(x, 1, 1, 1, x)$, and $H(x, 1, 1, 1, 1)$ (left panel), and $H(0, 0, 0, 0, x)$, $H(0, 0, 0, x, x)$, $H(0, 0, x, x, x)$, $H(0, x, x, x, 1)$, $H(0, 1, x, x, x)$, $H(0, x, 1, 1, x)$, and $H(0, 1, 1, 1, x)$ (right panel), as a function of $0 \leq x \leq 1$, as computed by 3VIL using the Runge-Kutta solution of the differential equations in t . In each case, the renormalization scale is $Q = 1$. At the endpoints $x = 0$ and $x = 1$, each of the values shown agrees with an analytic special case given in eqs. (5.52)-(5.61).

with automatic step-size adjustment. However, in some cases, the endpoint $t = 1$ is also a pole of one or more of the coefficients c_{ij} and c_i , even though all of the I , \overline{F} , G , and H integrals are well-defined there. (For example, this occurs if any of u, v, w, x, y, z vanishes.) In these cases, we need to use a somewhat unusual Runge-Kutta integration algorithm for the final step, such that there are no evaluations of coefficients at the final endpoint. We encountered a very similar problem in the case of TSIL, and here we employ exactly the same solution as described there, involving a particular choice of 5-stage, 4th-order Butcher coefficients. The reader is referred to ref. [33] for a more detailed description of this rather specialized Runge-Kutta strategy.

In special cases where the analytical values of one or more of the integrals is known, 3VIL automatically replaces the values obtained by Runge-Kutta by the results of the analytical formulas [or reduction of $G(0, u, v, y, z)$ to \overline{F} and I functions], using the results of section V. This is particularly useful because we find that the cases in which this is possible tend to be also cases in which the Runge-Kutta running is subject to relatively larger numerical errors.

For illustration,[‡] we show in Figure 7.2 the results for the integral H for selected one-parameter families of arguments, parameterized by a single variable squared mass $0 \leq x \leq 1$. The other non-zero squared mass arguments and the renormalization scale Q are chosen to be 1 in these examples. The values at the endpoints $x = 0$ and $x = 1$ are analytically known, and given in eqs. (5.52)-(5.61). Note that these integral functions vary smoothly with x , and tend to decrease as the squared mass arguments are increased.

[‡] The code used to obtain the data in this figure is included with the 3VIL distribution, as one example of how to use the software. Another provided sample user application program shows how to compute and extract all of the basis integrals for the case $(u, v, w, x, y, z) = (t, t, b, h, W, W)$ in the Standard Model, where particle names are used to represent squared masses.

We have also checked consistency of all of the other analytic special cases for I , E , F , \overline{F} , G , and H functions in section V, compared to the results obtained from Runge-Kutta integration of the differential equations in t . The results reported to the user by 3VIL are always the analytic ones, when they are available.

For input squared masses u, v, w, x, y, z and renormalization scale Q , 3VIL automatically evaluates simultaneously all of the basis functions H , G , \overline{F} , and I , and the associated functions E and F and I_ϵ , as well as the alternative basis bold functions (for those who may prefer them), **E**, **F**, **G**, and **H** in the conventions and notation given in section III above. The latter are evaluated and stored as the coefficients of ϵ^{-n} for $n = 0, 1, 2, 3$ ($n = 0, 1$ only for **H**). Utilities are provided in 3VIL for extracting the basis function values from the data struct after computation, for printing results, etc.

Although the integral functions are always real for non-negative squared mass arguments, they are computed and given as long double complex numbers. The magnitude of the imaginary part, which arises due to the Runge-Kutta integration off of the real axis in the complex t plane, therefore can serve as a check of accuracy of the calculation, as it should vanish in the idealized case of no computational error. Integration off of the real axis is not always necessary, and is avoided by default when possible, but if desired it can be forced by the user, and the magnitude of the deviation of the contour from the real t axis can be varied by the user, as a check. We find that the magnitude of the imaginary part computed by the Runge-Kutta method is often larger than the error in the real part (determined either by analytical evaluation when possible, or by varying the default characteristics of the integration), so we expect that the imaginary part is often a conservative error estimate.

For generic input parameters, the relative accuracy of the results is typically on the order of 10^{-9} or better, but it can be worse for difficult cases corresponding to pseudo-thresholds where some triplet of squared masses (x, y, z) of propagators meeting at a vertex have a small magnitude of $|\sqrt{x} \pm \sqrt{y} \pm \sqrt{z}|$. Even in the worst cases of H integrals with more than one such pseudo-threshold, the relative accuracy is typically about 10^{-4} or better, which should be good enough for practical applications at 3-loop order. For generic input parameters, the total computation time by 3VIL for the simultaneous computation of all of the integrals is well under 1 second on modern hardware, but it can be somewhat more for the especially difficult cases. For analytical cases, the computation time is extremely short and relatively negligible.

The `README.txt` file included with the 3VIL distribution available at [43] provides additional technical details regarding the numerical integration techniques employed, a complete description of the user application programming interface, and some sample user programs illustrating how to use the library.

VIII. OUTLOOK

In this paper, we have studied the basis functions for 3-loop vacuum Feynman integrals, and provided results and a public open-source software package, available at [43], to efficiently evaluate them. We plan to maintain, update, and improve the code package 3VIL indefinitely, and welcome suggestions and bug reports.

One obvious application of these results is to the computation of the effective potential (or

its derivatives) for a general theory, and for the Standard Model in particular, at full 3-loop order. At present, the Standard Model effective potential is known at 2-loop order [2], with 3-loop contributions known at leading order in QCD and top Yukawa couplings [45], including resummation of infrared-singular Goldstone boson contributions [46, 47] (see also [48–51] for further developments), and at 4-loop order at leading order in QCD [52]. Another possible application is to the computation of self-energy functions and higher point functions, for which the results of the present paper can be used in the limit of zero external momentum, or in systematic expansions in small external momentum. For example, in supersymmetry, loop corrections depend on a large number of distinct heavier superpartner masses. At the present time, the mass hierarchies of the superpartner sector are conjectural, at best, so that for the foreseeable future it seems most useful to present results in terms of basis functions that can then be evaluated numerically on demand.

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